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On the concept image of complex numbers

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On the concept image of complex numbers

A study of how Swedish students understand the concept of complex numbers was performed. A questionnaire was issued reflecting the student view of own perception. Obtained answers show a variety of concept images describing how students adopt the concept of complex numbers. These concept images are classified in four categories in order to clarify the learning situation. Furthermore, this study also revealed a variety of misconceptions regarding this concept, and most of the misconceptions were also possible to refer to the classification system. In addition, results from an identification test show that students have difficulties discerning the basic property of complex numbers, *i.e.*, that any number is a complex number.

Keywords: complex numbers, concept image, conceptions, misconceptions

1. Introduction

Mathematics is commonly associated with a high degree of abstraction, due to the large amount of theoretical concepts, methods, and models, sometimes being difficult to visualize and concretize. As a consequence, abstraction has been given a lot of interest in mathematics education (*e.g.*, [1-3]), and abstract thinking is considered to be a major goal of learning mathematics [4-7]. In fact, abstraction may be regarded as a tool to enhance the student's ability to reason and think [8], and is "a fundamental process in mathematics [...] and a basic step in the creation of new concepts" [9].

Although abstraction is crucial in learning mathematics, it can nevertheless cause obstacles in the learning. Hazzan proposed that reducing some abstraction would be an effective way for the students to "cope with the new concepts" in order to make the concepts "mentally accessible" and, thus, to facilitate the understanding of the concepts for the students [3, p. 75]. So, it is the personal understanding acquired by the students, with conceptions and possible misconceptions, that matters for the understanding of mathematics.

The expression "we see what we know" applies to several situations where the student does not necessarily see the same thing as the teacher or a researcher [10, p. 230]. This is confirmed by Marton and Booth stating that a concept can be experienced by students in several different ways [11]. Students either support a concept or reject it and their interpretation of this concept, as well as their capacity of understanding, can also be affected by their intuition [12-14]. The way the student adopts and interprets a mathematical concept also depends on influential factors that characterize students' thinking, like earlier achieved knowledge and experience as well as the situation where the concept occurs [15]. This means that the student visualizes the concept and creates a symbol or a mental model accordingly. In this respect, Heffer questions the validity of an "objective reality of mathematical concepts", suggesting that a given concept may have several different meanings [16, p. 100]. Depending on the receiver's conception, idea and interpretation, mathematical knowledge is a result of "our current conceptualization". Tall and Vinner claim that students often have preconceptions of a mathematical concept [17]. The problem occurring here is that the student hereby can make an unintentionally distorted interpretation. This is also confirmed by more recent reports, *e.g.*, Biza, Christou and Zachariades who describe students' perspectives and

thinking about tangent lines [15], Rösken and Rolka who analyze students' conceptual learning regarding the notion of the definite integral [18].

1.1 Theoretical Approach

The meaning of the term concept has been debated and a distinction between two concept qualities has been brought forward [17, 19-21]. In the present study, the model elaborated by Tall and Vinner is used, with the 'concept image' and 'concept definition' as a framework [17]. On one hand they introduce the formal definition as opposed to the subjective interpretation and individual reconstruction done by students. The term 'concept definition' is used to designate the formal mathematical definition that explains the concept: "a form of words used to specify that concept" [17, p. 152]. On the other hand, the term 'concept image' designates the subjective reconstruction of the concept and is defined as describing "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" [17, p. 152].

This means that the concept definition should be identical for everybody and does not change from one individual to another, whereas the concept image differs between individuals and is reflecting personal reconstructions of a definition which generates individual perceptions [17, 19-21]. Bingolbali and Monaghan clarify the situation with "while a tutor's definition of a concept may evoke correct associations for some students, many students will generate, amongst some intended associations, unintended *concept images*" [19, p. 19].

Based on this model with concept definition and concept image, Vinner tries to explain how and why misconceptions exist and why difficulties can occur in learning situations [22, p. 70]. During the concept formation the relationship between concept image and concept definition should be reciprocal and mutual. However, teachers tend to believe that during this process it is a one-way relationship, namely from the concept definition towards the concept image. Furthermore, Tall and Vinner mention the formation of a 'personal concept definition' which is not to be confused with the formal concept definition [17, p. 152]. The formal concept definition is the definition given and acknowledged by mathematicians, whereas the personal concept image is described as being the student's personal reconstruction of the definition or "the form of words that the student uses for his own explanation of his (evoked) concept image" [17, p. 152].

The formal concept definition of complex numbers is given in any textbook on the subject. Adams and Essex give the concept definition as: "A **complex number** is an expression of the form $a+bi$ or $a+ib$ where a and b are *real numbers*, and i is the imaginary unit" [23, p. A2]. Since the formal concept definition of complex numbers is quite undisputable, the present study will focus on the concept image as the framework basis for the analysis of the students' written responses.

Misconceptions can be of significant importance in learning situations, and traditional teaching is not known to eliminate misconceptions [24, p. 39]. Complex numbers can easily be associated with a variety of difficulties and misconceptions. The conceptions as well as misconceptions are often linked to the virtual nature of complex numbers and the terminology.

The main difficulty can be referred to the appellation itself. Students consider the words ‘complex’ and ‘complicated’ as synonymous, which generates an obstacle to learning. This confusion is due to the definition of the word ‘complex’ as “involving a lot of different but related parts, difficult to understand or find an answer to because of having many different parts” [25]. Furthermore, the word ‘imaginary’ has a kind of built-in negative connotation. This is also due to the definition of the word as “describes something that is created by and exists only in the mind” [26]. Such preconceptions can make the students feel that complex numbers are not practically useful. Thus, they have difficulties to consider complex numbers as something natural and given, but rather associate them with made-up quantities of a human invention without any relation to reality.

Earlier research regarding complex numbers has shown that students may experience difficulties for example when they have to switch between the different representations of complex numbers [27, 28]. Another major cognitive difficulty is that students are reluctant to accept complex numbers as numbers [29, 30]. The same attitude can be observed when young students are introduced to the rational numbers and they have difficulties accepting them as numbers [31]. Furthermore, Tirosh and Almog mean that students tend to incorrectly attribute properties to complex numbers which they do not possess, like ordering of numbers [29]. This difficulty, however, is not specific for complex numbers. Students tend to attribute properties belonging to natural numbers and incorrectly apply them to rational numbers as well [32-34].

The problem can be illustrated by an example from a Swedish high-school algebra lesson that caused some misconception. First-year high-school students (age 16-17) had difficulties in understanding why the expression $(x^2 - 1)$ can be factorized, whereas the expression $(x^2 + 1)$ cannot be factorized in the same way. The solution to this problem is not due until later in Swedish education (third year high-school mathematics, or even postponed to university mathematics). At this later stage, the teacher may introduce a new mathematical symbol, i , using the given identity $i^2 = -1$.

Inspired by the algebra lesson mentioned above, as well as the results of Conner et al. implying that students’ conception of the concept complex numbers often is limited to the expression $i = \sqrt{-1}$, [35], the focus of the present study is put on the concept of complex numbers and the way it is perceived by students.

The following research questions are fundamental for the present study:

- What concept images are created by the students when working with the complex numbers?
- Is it possible to classify students’ concept images in distinct categories?

2. Methodology

As a first approach a questionnaire was issued, and the answers to the questionnaire form the basis of the students’ conceptions and misconceptions of the concept of complex numbers. Our analysis is based on the answers obtained from the questionnaire. The results can be open to different interpretations. Nevertheless, it is possible to draw conclusions and to find a pattern in the answers.

The first questionnaire was released to 47 Swedish engineering students at university level. The choice of engineering students rather than mathematics students was intentional. The selection of students not having mathematics as the major subject, although important in the education, appears to be beneficial in order to pinpoint the misconceptions and identify the most common difficulties with complex numbers. After taking a one-week introductory course for engineers about complex numbers the students were supposed to have fundamental mathematical understanding of the concept. The knowledge gained should be used directly in an engineering course of electric-circuit theory where the use and manipulation of complex numbers are essential in direct application.

This part of the study is a qualitative approach to get a picture of student concept images of complex numbers. The limited number of participants is not statistically sufficient, but it is rather the type of information given in the answers than the quantity being evaluated and analyzed.

The questionnaire was designed in order to provide the students with an opportunity to express thoughts and conceptions of the subject. The questionnaire was given in Swedish, but here it is translated to English, see Figure 1.

Figure 1. Questionnaire to university engineering students.

- *What do you connect to the concept complex numbers?*
- *What is the greatest difficulty you experience regarding complex numbers?*
- *Can you give some examples of application for complex numbers?*
- *Fill the mind map below with words that best describe your conception of complex numbers. What do complex numbers evoke in your mind?*

The questions were entirely open and not given as mathematical tasks. This was intended as a means to make the students express their feelings and views about complex numbers. Participation in the study was completely voluntary. The students were informed that this is not a test included in their education and that the results will not be graded, but only be used in research purposes. Furthermore, they were informed that there are no correct or incorrect answers. The answers were given anonymously and they got 15 minutes to answer all the questions, since the answers were not supposed to be highly elaborated.

The intention with this part of the investigation was to pinpoint if categories of complex number concept images would crystallize from the answers given by the students. A questionnaire form was preferred rather than personal interviews due to the quite large amount of students who participated in the study. The answers were sorted, analysed and categorized into four categories.

In order to check how achieved concept images relate to the formal concept definition, a simple test was performed with a quite large number of students during 5 minutes. In order to check if the maturity level can be of significance here, the authors chose to conduct this test on two different populations: students from high school and students from university who both were taking a course with complex numbers. So, in this test, 31 Swedish university engineering students and 31 senior high-school students answered the question “Which of these are complex numbers?” (here translated to English), see Figure 2.

Figure 2. Identification test “Which of these are complex numbers?”.

a) i	Yes ☐	No ☐
b) i^i	Yes ☐	No ☐
c) $-2,5$	Yes ☐	No ☐
d) $e^{i\pi}$	Yes ☐	No ☐
e) $5\cos \pi$	Yes ☐	No ☐
f) $\cos \pi + i \sin \pi$	Yes ☐	No ☐
g) $\cos \pi + \sin \pi$	Yes ☐	No ☐

3. Results

3.1 Students' conception of complex numbers

47 Swedish engineering students were invited to answer a questionnaire (see Figure 1), in order to elucidate their concept image of complex numbers after an introductory course on the subject. It is important to note that the questionnaire, as well as the answers to the questionnaire, were given in Swedish. Thus, all the answers and the results presented here are translated from Swedish to English. The authors were careful in making the translated answers as accurate as possible in reflecting the original answers.

In the following, some ways of how the students conceive complex numbers are categorized. Some students did not answer certain questions and some other students gave multiple answers, which explains that the total amount of answers do not agree with the total amount of students.

- 15 students conceive complex numbers as a way to make some calculations possible through the expansion of the set of real numbers. Some of those 15 students also specified that it is a way of solving negative square roots.
- 10 students conceive complex numbers as two-dimensional numbers consisting of a real part and an imaginary part.
- 7 students associate the complex numbers with the identity $i^2 = -1$, or simply with the symbol i .
- 11 students gave more emotional or negative answers, *e.g.*, “abstract”, “complicated”, “difficult”, “complex, which explains the name”, “no conception”, and “don’t know”.
- 4 students did not answer this question.
- 3 students gave irrelevant answers.

In the following, blank answers as well as erroneous or irrelevant answers are disregarded. As a result of all the data obtained from the analysis of the answers to the questionnaire, four categories of complex-number concept images are obtained, *viz.* a mathematical artifice, two-dimensional numbers, a symbolic extension of mathematics, and an ungraspable mystery. The listing is done without any aspect of validity of the authors, since the categories are reflecting the learning situation of the students.

The authors do not claim that these categories are unique, *i.e.*, there may be other classification systems, but in this investigation the results support these classification categories. The classification system also has support in literature.

3.1.1 A mathematical artifice

This category is symbolizing Sfard’s description of the development of complex numbers as an extension of the real numbers, due to the need for additional objects with new properties [36, p. 12].

When complex numbers are viewed as a mathematical artifice, the student’s concept image is not associated with reality. The concept image is reduced to a human invention to facilitate calculations and force solutions to exist in equations that otherwise would not be solvable. Feelings of a mathematical concept as being a man-made artifice may generate a negative attitude to the subject, and block the student from further development and understanding of the concept. It is obvious that if the student cannot believe in a concept and its authenticity, then it will be practically impossible to comprehend and grasp this concept, as well as to use it as a fundament for further education. This is also confirmed by Pehkonen [37].

3.1.2 Two-dimensional numbers, or the two-dimensional view

In this category, the students are considering the ‘complex’ nature of complex numbers. This is illustrated by CDO defining the word “complex” as “involving a lot of different but related parts” [25]. With this concept image, the students are only able to see the complex numbers as two separate entities. A complex number should be conceptualized as one number, *i.e.*, the expression $a+ib$ is a single entity combining a real number and an imaginary number. But when the students give answers such as “two components”, they are failing to see the complex numbers in this way.

3.1.3 A symbolic extension of mathematics, or the symbolic view

In this algebraic way of regarding and conceiving the nature of the complex numbers, the concept image is based on the symbol i^2 or simply the symbol i . The primary

function of symbols is to “represent something else” [38]. Thus, this type of answers, which are reducing the concept image of complex numbers to a symbolic expression, may be interpreted as an attempt of the students to reduce the abstraction of the complex numbers and to envision them in a more concrete way. Jourdain was among the first to illustrate this by stating that mathematicians were insecure about writing $\sqrt{-1}$, so they used i instead and considered the symbol i as merely operational [39, p. 38].

3.1.4 An ungraspable mystery, or the mystery view

Students classifying the complex numbers as an ungraspable mystery are shifting their attention from the cognitive aspect to the emotional aspect. Unconsciously, the following feelings and beliefs are expressed: the abstract and theoretical nature of the complex numbers, and possibly the lack of visualization, constitute an obstacle in understanding the use of the concept. In fact, Arcavi points out the importance of visualization in the learning of mathematics, in order to create an adequate concept image [10]. Hazzan claims that due to abstraction, students may “fail in constructing mental objects for the new ideas and in assimilating them with their existing knowledge” [3, p. 84]. In this category of concept images, the students will question if those numbers are valid ‘in real life’. Thus, it is reasonable to assume that concept images based on a mystery view will differ between individuals.

Judging from the students’ answers to the second question in Fig. 1, it seems that the most severe dilemma of getting the concept image to match the concept definition is related to abstraction. Considering that 16 students did not answer this question at all, only 31 answers are taken into consideration.

- 10 students are explicitly stating that they are experiencing the complex numbers as a difficult concept, due to the fact that they cannot *visualize* them. These students are also questioning the use of complex numbers ‘in reality’, and mention the difficulty to imagine what complex numbers ‘stand for and really are’.
- 6 students have difficulties in memorizing all the rules and formulas associated with complex numbers.
- 6 students mentioned difficulties in managing the polar/trigonometric representation of a complex number.
- A small part of the students expressed more specific aspects of difficulties, *e.g.*, to identify the real part and the imaginary part of a complex number, to apply the complex numbers to real situations in life, and to understand the complex plane.

The students appear to be confused when asked to give examples illustrating the application of complex numbers, see the third question in Fig.1. More than half of the students (28 out of 47) had absolutely no idea of how complex numbers are applied and can be used in ‘real life’.

3.2 Students’ answers to the mind map

In the results of the mind map (the last question in Fig. 1), a wide range of ideas and conceptions of complex numbers was given by the students. Examples of answers in the mind map are given below. In brackets the category of concept image is indicated by the authors if applicable.

- Real part and imaginary part (15 answers) (The two-dimensional view)
- $i^2 = -1$ (13 answers) (The symbolic view)
- Two components (11 answers) (The two-dimensional view)
- i or even i^2 (11 answers) (The symbolic view)
- Difficult or even complicated (9 answers)
- Theory and abstraction (6 answers) (The mystery view)
- Different representations (6 answers) (The symbolic view)
- cos and sin (3 answers) (The two-dimensional view)
- $\sqrt{-1}$ (3 answers) (The symbolic view, or even the mystery view)
- Argument (3 answers)
- Lack of usefulness and when to use? (2 answers) (The mystery view)
- Electronics (2 answers)
- Vectors (2 answers) (The two-dimensional view)
- Conjugate (2 answers)
- Complex plane (1 answer) (The two-dimensional view)
- Geometry, tough, boring, not logical, new, $|z|$, interesting, time consuming, etc. (1 answer per student) (The mystery view, combined with emotions of current stage of learning)

Other answers were also obtained through this mind map. A lot of them were judged irrelevant to the context of this study, *e.g.*, factorizing, infinite, quadrant, rules, Pi, number line etc.

3.3 Students' answers to the identification test of complex numbers

In order to check how achieved concept images relate to the formal concept definition, a simple test was performed with a quite large number of students. In this test, 31 Swedish university engineering students and 31 Swedish senior high-school students answered the question "Which of these are complex numbers?". The time given to answer the test was approximately 5 minutes.

Table 1. Results from the identification test performed by 62 students (31 university engineering students and 31 senior high-school students).

Which of these are complex numbers?

Task	Value	Frequency		Discrepancy
		Yes	No	
a) i	i	57	5	
b) i^i	$e^{-\pi/2}$	48	12	2 missing answers
c) -2.5	-2.5	13	52	3 multiple answers
d) $e^{i\pi}$	-1	50	11	1 missing answer
e) $5\cos \pi$	-5	13	51	2 multiple answers
f) $\cos \pi + i \sin \pi$	-1	54	8	
g) $\cos \pi + \sin \pi$	-1	11	53	2 multiple answers

In Table 1, the answers from all 62 students are given. The headings in Table 1 are 'Task', *i.e.*, the expression shown to the students to consider as a complex number or not, 'Value', *i.e.*, the calculated result if elaborating on the given number, 'Frequency', *i.e.*, the number of Yes and No answers, respectively, and 'Discrepancy', *i.e.*, comments on answers missing or multiple answers.

The students did not have the column “value” shown when they answered this question (see Fig. 1). It has been added by the authors for the purposes of this study. It is interesting to calculate this value in order to understand how the students were thinking when they answered this test. An important observation is if the explicit presence of the imaginary unit, i , would influence their answers.

No significant difference between university engineering students and senior high-school students were found.

- 92% of the students (57 out of 62 students) answered that i is a complex number, which is the majority of the participants regardless of their age or maturity level.
- 77% of the students (48 out of 62 students) answered that i^i is a complex number, although the calculated value is a real number.
- 84% of the students (52 of 62 students) answered that -2.5 is **not** a complex number.
- 81% of the students (50 out of 62 students) answered that $e^{i\pi}$ is a complex number, although the calculated value is a real number.
- 82% of the students (51 out of 62 students) answered that $5\cos\pi$ is **not** a complex number.
- 87% of the students (54 out of 62 students) answered that $\cos\pi+i\sin\pi$ is a complex number, although the calculated value is a real number.
- 86% of the students (53 out of 62 students) answered that $\cos\pi+\sin\pi$ is **not** a complex number.

Some students either did not answer or gave multiple answers to some of the tasks a) - g), thus giving frequency discrepancies in Table 1. Only 7 students out of 62 (2 university students and 5 senior high-school students) gave correct answers to all tasks. However, one of the university students gave multiple answers for the options he immediately realized being real numbers, *i.e.*, c), e), and g) and motivated this by writing “could be with $0i$ ”.

4. Discussion

It is particularly difficult to estimate and analyze students’ conceptions and misconceptions in mathematics. This is partly due to the complicated task of making examination tests that actually measure the students’ knowledge in the area, and not only mechanical skills. Mathematics is a subject often examined by means of skill questions, *e.g.*, ‘solve this equation’. Questions measuring conception and understanding are less common and may be difficult to find and formulate.

Indeed, it is very difficult to imagine or try to guess “what is happening inside the head of a student”. The question here is if the teachers are aware of students’ thinking, weaknesses, and misconceptions. Hence, from a pedagogical viewpoint the problems should be possible to avoid. Teachers’ awareness of students’ misconceptions should “contribute to the improvement of their teaching” [24, p. 39]. But does it really?

4.1 Students' conception of complex numbers

By means of the answers of the questionnaire given to 47 Swedish university engineering students four categories crystallized, summing up the obtained answers to the questionnaire. These categories, detailed in paragraph 3.1, are

- a mathematical artifice
- two-dimensional numbers or the two-dimensional view
- a symbolic extension of mathematics or the symbolic view
- an ungraspable mystery or the mystery view.

There is a clear connection between the need of visualization for learning and three of the previously defined concept images, *viz.*, the mystery view, a mathematical artifice, and the symbolic view. This means that students consider new concepts which they cannot visualize as mysterious and/or symbolic, or even as man-made artifices without sense in reality. This attitude is clearly an obstacle to learning!

Another problem showing up was the fact that more than half of the students (28 out of 47) had absolutely no idea of how complex numbers are applied and can be used in 'real life'. This emphasizes the importance of a deep approach to learning where students are encouraged to associate knowledge from different courses, relate previous knowledge to new knowledge, and connect theoretical concepts with everyday experience [40, 41]. Some examples taken from science or technical subjects may thus serve as a door-opener for the mathematics education, providing that the examples are appreciated by the students as relevant and important in their professional career. As an example to illustrate this, electric-circuit theory is often considered as abstract although laboratory work should give some justification to theories gained, and, consequently, to the mathematical fundament on which the theories are based. However, there will always be kind of a Catch-22 situation in such a learning scheme, and maybe the electric-circuit laboratory work (or visual real-life tasks) should be introduced even before complex numbers are studied in a mathematical context.

4.2 Students' answers to the mind map

The results from the mind-map exercise are to a high degree consistent with the categories of concept images found. The answers which are not possible to categorize in this classification system express frustration of not having total mathematical control over the calculations involved in complex numbers. Statements like Difficult, Complicated, Argument, Conjugate, Geometry, Tough, Boring, Not logical, New, $|z|$, and Time consuming, are expressing emotions relating to the current stage of learning, and may support a future mystery view if nothing is done to influence the concept image. Statements like Interesting and Electronics, on the other hand, show some anticipation of future use.

4.3 Students' answers to the identification test of complex numbers

It seems in the identification test that students indeed connect complex numbers with two-dimensional numbers. This confirms that the two-dimensional view dominates here. However, if students cannot see the imaginary unit i explicitly, they consider the

number not to be a complex number. This is shown in Figure 3 which is a graphical representation of the results presented in Table 1. In Figure 3, the tasks for the question "Which of these are complex numbers?" are again given for clarity.

Figure 3. Results from the identification test taken from Table 1.

In tasks a), b), d) and f) the imaginary unit is easily found. The short time for the students to take this test (approximately 5 minutes) did not give the possibility to realize that all these numbers, except a), are real numbers, see Table 1. If the students had a fair chance to elaborate on the numbers the result may have turned out differently. In tasks c), e), and g), the students did clearly see that the numbers are real numbers since the imaginary part is missing. The trend is obvious – in the two-dimensional view a complex number must have two parts, a real part and an imaginary part. Or at least the imaginary unit must be visible. This phenomenon is also confirmed by Tall stating that students consider the real number $\sqrt{2}$ as not being a complex number, while they consider the number $\sqrt{2} + i0$ to be a complex number [42, p. 4].

Only 7 students out of 62 (2 university students and 5 senior high-school students) gave a totally correct answer. Some of them gave a remark on their judgment, maybe because they had an awkward feeling about it. Remarks like “All numbers are real numbers” or “All numbers that can be drawn in the plane are complex” were found. On the other hand, one of the university students gave multiple answers for the options he immediately realized being real numbers, *i.e.*, c), e), and g) and motivated this by writing “could be with $0i$ ” in agreement with the report of Tall [42].

One of the mathematically most talented senior high-school students claimed that none of the tasks represented complex numbers. In a personal interview he explained his standpoint by saying that since there was only one part in all the tasks (either the real part or the imaginary part was missing) they could not be complex numbers. Obviously his skills in mathematical manipulation showed that the numbers in all of the tasks (or at least a majority) were either real or purely imaginary. For him a complex number comprises two parts in accordance with the results in other studies (*e.g.*, [42]), which means that his concept image is the two-dimensional view.

In fact, the two-dimensional view corresponds to the formal concept definition of complex numbers which states that a complex number is an expression of the form $a+bi$ or $a+ib$. But just to apply the formal concept definition literally did not prevent the students from having a distorted concept image of complex numbers. It is not sufficient to just apply the concept definition without a correct interpretation or a correct

constructed concept image, because most of the students failed to see that all the numbers in the table are complex numbers, especially in the case where $b=0$. This phenomenon of taking the concept definition and varying it from time to time is called the student's personal concept definition. The personal concept definition can differ from a formal concept definition, but it is only the formal concept definition that is accepted by the mathematical community [17, p. 152]. As shown in the results of the identification test, students may have altered the concept definition and transformed it into a personal concept definition where the presence of the imaginary unit is mandatory and thus failed to answer the tasks correctly.

5. Conclusion

The aim of this study is to contribute to an understanding of mathematical learning. When teaching the concept definition and processes descending from this definition, the teacher should be aware of the variety of concept images evoked in the students' minds.

Associating words like 'complex' and 'imaginary' to this rather abstract theme of mathematics is harmful for learning. Perhaps the term 'complex numbers' should be replaced by a more appealing nomenclature for pedagogical success.

Returning to the research questions about the concept images being created by the students, it is possible to state that students have a rich and wide spectrum of concept images as well as diversified associations when dealing with complex numbers. Nevertheless, it was possible to delimit those associations and gather them into categories.

The main contribution of the present study to the learning of complex numbers is a simple model consisting of 4 categories of concept images, *viz.* a mathematical artifice, two-dimensional numbers, a symbolic extension of mathematics, and an ungraspable mystery. This set of categories is not claimed to be unique, and possibly other sets of classification may be found. However, the classification system can be applied as a tool for identifying, describing, and further understanding students' thinking, conceptions and misconceptions.

From a didactical standpoint, an interesting result is that the two most frequently evoked concept images of complex numbers are diametrically opposed:

- the two-dimensional view, concluded by the most frequent answers from the mind map and the identification test.
- the mystery view, concluded by the most frequent answers from the difficulties students are experiencing when working with complex numbers.

The identification test together with the results obtained by the mind map, show that the two-dimensional view is dominating the thinking of complex numbers. On the other hand, the two-dimensional view is regarded by the authors as the category relating the closest to the formal concept definition, although it is shown that the student personal concept definition in this case may exclude numbers from being complex numbers.

This study has shown that the most common difficulties students are facing emanate from the theoretically abstract property of complex numbers. Students can get a quite

distorted concept image. Most of the issues in the study - conception, misconceptions, reason behind difficulties - are closely related to the abstract nature of complex numbers. Learning an abstract mathematical concept may be a motivating challenge for some interested students. However, in the majority of cases, with average students in an average classroom situation, lack of visualization can be expected to generate severe obstacles to learning. A successful way of teaching complex numbers is probably depending on the teacher's capability of leaving behind the classical ways of introducing the concept. To introduce the concept with an innovative and visual method could be a way to enhance the teacher's professional growth as well as a better learning situation for the student.

In a coming study, methods of introducing complex numbers in order to reduce the feeling of 'complexity' are investigated. The objective is to make the students' concept images move from the mystery view and approach the concept definition. A method of introducing complex numbers in a visual way with a feeling of less abstraction for the average student could be the key to pedagogical success in this area. Such an approach would most certainly also strengthen the concept image of the mathematically talented students.

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