IDENTIFYING ALGEBRAIC REASONING ABOUT FRACTIONS

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The issue for this paper is to identify algebraic reasoning through students´ sense-making actions, during a lesson, where students and a teacher develop learning models for mixed numbers. The analysis focuses the students' work, trying to make sense of the unknown fractional part of the number. This unknown part was elaborated when the students suggested to "add a little bit more" to construct equality. The unknown part developed to a fractional part with help of an emerging learning model containing algebraic symbols: B=W+p/a. In this activity, I identified potentialities in the students' algebraic reasoning; an additive relationship between the integer and the fractional part of the number, and a multiplicative relationship between the numerator and the denominator in this fractional part.

INTRODUCTION

There is an assumption that mathematical education for the youngest students can be organized in relation to an algebraic tradition described by van Oers (2001). In this tradition, problem solving and algebraic reasoning are essential for the development of mathematical thinking. Davydov's (2008) suggests that an algebraic tradition can be designed through the theoretical framework–learning activitydeveloped to emerge students' theoretical thoughts, in object-related processes, in a practical, cultural–historical activity. In the research field of early algebraization, Davydov's ideas are frequently referred to as means to develop algebraic reasoning, jointly with young students (Cai & Knuth, 2011). In this research field, the underlying purpose of developing algebraic reasoning is to deepen children's understanding of the structural forms, relations, and generalities in mathematics (Cai & Knuth, 2011). However, certain questions in this research field still require answers (Blanton & Kaput, 2011; Nunes, Bryant, & Watson, 2009; Lee & Hackenberg, 2013). Considering their questions, I discuss what can be identified as an emerging algebraic reasoning about fractions: What indicates algebraic reasoning, identified through students' sense-making actions, when the students are analysing quantities?

THEORETICAL BACKGROUND

The concept of objectification

In the algebraic tradition, learning is seen as object-related processes, as processes of objectification, by which students gradually become acquainted with historically constituted cultural meanings and forms of reasoning and actions (Davydov, 2008;

2018. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education,* (Vol. 1, pp. 255-262). Umeå, Sweden: PME.

Radford, 2015). These processes are rooted in a dialectical materialistic conception of humans and the world (Leontiev, 1978; Radford, 2015). Following on that, objectification posits an object and the thought about the object as heterogeneous entities (Radford, 2015). In the process of objectification, an object of knowledge can be seen as opportunities for actions, for example, opportunities to calculate. This means that objects of knowledge cannot be accessed directly, they are always mediated by activity, by sense-making actions. Furthermore, according to Radford (2015), the conception of objectification is based on a distinction between two related categories in an object of knowledge: the potentiality and the actuality. Potentiality can be understood as the capacity, ability, or power to do something. Objects of knowledge belonging to this category are treated as undeveloped, lacking in connections with other things, poor in content, and formal in that the object is not yet connected to any concrete examples. Actuality, on the other hand, is described as "being-at-work", i.e., that something in motion occurring in front of us or through our work. Between the potentiality and the actuality are sense-making actions. So, knowledge and its objects have three elements: their potential, their actualization, and the sens-making actions in the activity that mediates them. A motion of an object can appear through the contradictions when the object is realized through sense-making actions in an activity. The process by which the object is set in motion, the process that changes the object from its potentiality to its actuality through sense-making actions, can be described as ascending from the abstract to the concrete (Davydov, 2008; Radford, 2015).

Algebraic reasoning

In the algebraic tradition, algebraic reasoning as an aspect of algebraic thinking, is explained as the student's ability to: 1) understand patterns, relations, and functions; 2) represent and analyse mathematical situations and structures using algebraic symbols; 3) use mathematical models to represent and understand quantitative relationships; and 4) analyse changes in various contexts (Cai & Knuth, 2011). More specific explanations of algebraic thinking are offered by, scholars such as, Davydov (2008, p. 148) and Radford (2015; 2018). Davydov is focusing on relations in, and between, quantities. He suggests that algebraic thinking entails: 1) an introduction to relations between quantities; 2) discovery of the ratio relation in quantities; 3) the formation of all real numbers; and 4) discovery that any mathematical operation has a structure. Radford suggests that algebraic thinking entails: 1) reasoning about indeterminacy; 2) denotation of this indeterminacy using natural language, gestures, unconventional signs, or a mixture of these; and 3) analyticity in which indeterminate quantities are treated as if they were known numbers. These explanations are broader than the traditional explanation of algebra as simply constituting general arithmetic. Using Davydov's and Radford's explanations, algebraic thinking can be seen as opportunities for students to analytically deal with unknown quantities and unknown numbers in relation to different objects of knowledge.

For these analyses, undertaken by even the youngest students, Radford (2014) suggests a classification of three forms of algebraic thinking: non-symbolic, contextual, and symbolic. In *non-symbolic algebraic thinking*, students analyse, for example, an unknown quantity, using words from everyday life and argue for ways to implicitly solve a mathematical problem, using, for example, gestures and rhythms (Radford, 2014). Here, students can reason analytically and operate with unknowns without using numerical symbols, with the help of gestures and words they usually use. Students' *contextual algebraic thinking* comprises symbols and rhythms in relation to an explicit subject content. Finally, students' *symbolic algebra thinking* involves alphanumeric formulas. All three types of algebraic thinking are regarded as vivid narratives instead of just calculations using formulas (Radford, 2014); they can be used as descriptors to identify the sense-making actions students do during their analytical work of unknown quantities and the relationships in and between these quantities.

Fractions

Difficulties and misconceptions in relation to developing number sense in regard to fractions have been presented in many previous studies (e.g., Davydov & Tsvetkovich, 1991; Nunes et al., 2009). Davydov (2008) argues that number senses regarding whole numbers and rational numbers should develop out of the same context and cultural–historical tradition of numbers. Thus, comparing quantities is rooted in human cultural–historical work, and accordingly, comparing and measuring are important activities in developing a number sense (Davydov, 2008). To enhance number sense regarding fractions that arises out of measuring, Davydov and Tsvetkovich describe three key concepts to be addressed: a quantity to measure (e.g., quantity as a length, weight, volume, or area), a unit of measure, and a smaller unit with which to measure the unit of measure. For whole numbers, the quantity to measure can be described using an integer, while for rational numbers the quantity must be described using an integer and a remainder, with the remainder being described in terms of the relation between the numerator and denominator.

DATA AND METHODS

The data used here consist of transcripts from one lesson in a research project to develop students' understanding of fractions (Eriksson, 2015). At the core of the project was a collective iterative process (involving me as a researcher and one of the three participating teachers) of designing and redesigning a learning activity. The twenty participating students were 9, 10, and 11 years old, more than half of them were newly arrived in Sweden. The lesson drew on the so-called El´konin–Davydovs' mathematical curriculum with learning activity as a framework (Davydov, 2008). The actual learning tasks in the lesson were inspired by Davydov and Tsvetkovich (1991).

To develop a learning activity, it is important to start with a common problem that the students and the teacher can collaboratively identify. In a "pre-lesson" before the research lesson, the students first used the measurement activity to represent equalities using Cuisenaire rods. These equalities were presented in the form $A = WB$, where rod A could be described by a whole number (W) of rods (B). When the students were asked to design their own equalities, in which the length of one rod equalled the length of a whole number of just one other kind of rod, the students faced the problem that the length of rod A could not equal any whole number of any other kind of rod. One of the measures constructed by the students in this pre-lesson was to compare a black rod with red rods (Fig. 1), while another was to compare a black rod with yellow rods (Fig. 2).

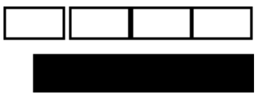

Fig. 1: Black = 3 red + ½ red

Fig. 2: Black = 1 yellow + 2/5 yellow

To find the quantity of the red and the yellow rods, the problem was to identify a fractional part of these rods. To solve this problem, the students had to identify the length to be measured (i.e., a black rod), the unit of measure (i.e., the red or yellow rod), and the quantity of this unit. The result had to be presented in the form $A = \mathrm{W}B + rem$, where A is the length to be measured, B is the unit of measure, *rem* is the remainder when constructing the equality, and W is the number of whole units of measure (compare Davydov & Tsvetkovich, 1991).

A first step in the analysis was to describe the activity as a narrative description suggested by Eriksson (2017). This description was done in relation to fractions while the students and the teacher were measuring a black rod in terms of red or yellow rods (see Figs. 1 and 2). A second step in the analysis was to identify the potentiality of the object of knowledge and its actualities, as suggested by Radford (2015; 2018). Four central sequences of the transit from potentiality to actuality were identified in the narrative description, coded 1, 2, 3, and 4. Analytical questions asked in this step were: What changes of the object of knowledge can be identified? and, What sense making actions made these changes possible? The last step of the analysis focused the sense making actions in these sequences. Analytical questions asked in regard to the sense-making actions were: What actions can be identified as an emerging algebraic reasoning described by Radford (2015)? and, How, was the unknown fractional part elaborated?

ANALYSES

This section presents the analysis of the four identified central sequences, which are illustrated using excerpts and observations from the lesson.

Sequence 1:

When the students entered the classroom, one of the measurements from the pre-lesson was attached to the whiteboard (Fig. 1). As the teacher drew a number line on the whiteboard; the students reacted by telling the teacher that they remembered the problem with that measurement.

> Teacher: Then, what was the problem? Do you remember what your problem was?
> [After a short whole-class discussion]
> Bayar: It is three and a half. There is three of these, and then there was one rod that was longer, so if you add one more it would be longer than the black rod. But if you just add a little bit, they would be the same length. If you add one half.
> [While saying this, Bayar was sitting at his desk, pointing at the depicted measurement on the whiteboard. First he moved his hand in three distinct gestures, pointing at the three whole units (one, two, three). Then he moved his hand up and down in small movements, and at the end of his comment, he put his other hand in the middle of the up and down movement.]

In this sequence, the students changed the object of knowledge from being just a measurement problem involving two lengths that represented inequality, to three whole units and one unit "that was longer". The sense-making constituting this change was conveyed by Bayar, who first pointed at the three whole units (one, two, three), saying: "There are three of these". Then, he indicated that something was different about the fourth unit (moving his hand up and down, and putting his other hand in the middle of the gesture), saying: "If you just add a little bit, they would be the same length". In this sequence, Bayar and the other students also indicated that there was an additive relationship between the whole and fractional part ("if you just *add* a little bit"). Altogether, these actions can be identified as illustrating emerging algebraic reasoning in relation to the students' analysis of the unknown quantity in the measurement problem. A reasoning expressed using non-symbolic means, through natural language and gestures without using numerical symbols to refer to the fractional part. The students were initiating theoretical work about the abstract additive relationship involving a mixed number ("add just a little bit"). The teacher used this as a resource to further develop the students' theoretical work about fractions.

Sequence 2:

> [The whole-class discussion continues.]
> Teacher: How can you know that? How can you know that you have to add a half?
> Leart: Maybe, we can measure? We do need to measure the red rod, don't we? And the little bit left?

In this second sequence, the object of knowledge was changed, or developed, from the suggestion "add a little bit" to the further analysis of the fractional part, i.e. that they had to measure the unit of measure in order to understand the fractional part. The emerging algebraic reasoning was identified as a further step in analysing the unknown fractional part, in which measuring even the unit of measure was suggested.

Sequence 3:

Teacher: We have to return to our problem. What are we going to do?
Dana: We are going to tell the black …
Chaid: The black is the whole one and there, it is a little bit more.
Evin: Last lesson, this was our problem …
Ami: We need something that is smaller than the red ones to measure with.
[When Ami suggests this, the teacher takes two white rods, a shorter unit than the red rod, and puts them on the board beside the last red unit.]

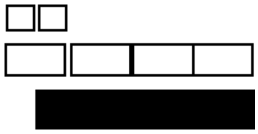

Fig. 3: The shorter (white) unit was put in the measurement on the white board.

In this third sequence, the object of knowledge was changed, and further developed, from a need to measure, to a need for a unit shorter than the original unit of measure. The actions that made sense of that change were Ami's suggestion to use a smaller unit and the teacher's modification of the measure (Fig. 3). The emerging algebraic reasoning can thus be identified as the analysis of the unknown fraction part of the original measurement unit with the new shorter unit.

Sequence 4:

[The teacher initiated a discussion of how to represent this number, starting with the students' comment "a little bit more", which was used as an emerging model of the fractional part of the number. The teacher wrote: "*Black* = W + *a little bit more*" on the board. A further step of the learning model, comprised of the idea to use a shorter unit to measure the unit of measure, was written by a student: *B* = W + *white rod*/*white rods*. A third learning model was then suggested and written by the teacher: *B* = W + m/n. This model was rejected by the students, who instead suggested: *B* = W + p/a. According to the students, *W* is the whole unit, *p* is the part of the last small unit needed to achieve equality, and *a* is all the small units used to measure the original unit of measure. Next, measuring the black rod with yellow rods (see Fig. 2), the students developed the learning model: *B* = W*y* + (p/a)*y*. The final learning model was realized to be even more connected to the empirical measure. The students suggested: *Black* = W*yellow* + (p/a)*yellow*.]

Through these four sequences, the object of knowledge underwent a stepwise emergence of a learning model containing algebraic symbols. In the first step (*Black* = W + *a little bit more*), the actuality object of knowledge initiated by the students was the additive relationship between the integer "W" and the fractional part of the number implied by "*add a little bit more*". In the second step (*B* = W + *white rod*/*white rods*), the relationship between the numerator and the denominator emerged. The actuality object of knowledge in this step was the multiplicative relationship between the numerator and denominator in the fractional part of the number. In the third step of the model (*B* = W + m/n), the symbols "m" and "n" were suggested as they are common algebraic symbols for the numerator and denominator (according to the teacher group that planned this lesson). However, in the fourth step (B = W + p/a), the students instead wanted to use algebraic symbols with a contextual meaning, i.e., "p" for the partial unit they needed to develop equality and "a" for all the small pieces into which the unit of measure had to be divided into. Here, the students used symbols in an algebraic way, to represent relationship between variables in a general way. By doing so, they connected the symbols to the context of the fractional part of the number, "p" (i.e., numerator), and "a" (i.e., denominator). These symbols emerged from the multiplicative relationship in the fractional part of the number. As suggested by the students, the learning model then became increasingly concrete, i.e., increasingly connected to the specific measures addressed, as they were embodied by the rods. Furthermore, in measuring a black rod using yellow rods, the students first suggested the learning model B = W*y* + (p/a)*y* and then *Black* = W*yellow* + (p/a)*yellow*. The students' algebraic reasoning can thus be identified as the analysis of the additive relation between the integer and the unknown fractional part of the number, and as the multiplicative relationship in the unknown fractional part.

RESULTS AND CONCLUDING REMARKS

In this paper, algebraic reasoning about fractions was identified through sense-making actions when the students and the teacher analysed the unknown fractional part of mixed numbers (Davydov, 2008; Radford, 2014). These sense-making actions focused on different relationships in a mixed number, such as an additive relationship between the integer and the fractional part, and a multiplicative relationship in the fractional part of the number (Davydov, 2008). The fractional part of the number was first analysed using natural language (e.g., "add a little bit") and gestures when the students and the teacher were pointing at the various Cuisenaire rods. Such sense-making, according to Radford (2015), can be described as non-symbolic algebraic thinking. I noticed how the students identified the additive relationship between the integer and the fractional part of the number (Davydov, 2008; Davydov & Tsvetkovich, 1991), and how, through further analysis of the unknown fractional part, the students suggested that a smaller unit should be used to measure the unit of measure (Davydov & Tsvetkovich, 1991). This suggestion, to measure the unit of measure, made sense to the students as they were exploring the multiplicative relationship inherent in the fractional part of the number, which was further developed when a student wrote "one rod/two rods" on the board. The learning model for the unknown fractional part, containing algebraic symbols emerged from, *B* = *W* + *a little bit more*, leading to the more concrete model connected to a specific measure, *Black* = *W*yellow + (*p*/*a*)yellow. Following Davydov (2008) and Leontiev (1978), we can describe the students' elaborations of the learning model as processes of objectification; as ascending from the abstract to the concrete. The students' suggestions of what algebraic symbols to use in contextualizing their algebraic reasoning manifest Radford's (2015; 2018) descriptions of contextual algebraic thinking. In this lesson, the object of knowledge—the unknown fractional part of the number—emerged through a process of realizing the potentiality of the object (i.e., add a little bit more), through sense-making actions (i.e., algebraic reasoning in a contextual manner), to various actualities of the object.

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