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Lifting the understanding of trigonometric limits from procedural towards conceptual

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ABSTRACT
The purpose of this paper is to follow the reasoning of high school students when asked to explain the standard trigonometric limit
\[ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1. \]
An observational study was conducted in four different phases in order to investigate if visualization, by means of an interactive technology environment (GeoGebra), can contribute in lifting high school students’ understanding from a mere procedural understanding to a combination of conceptual and procedural understanding. The obtained results confirm that the students were able to show a conceptual understanding only after using the digital interactive tool. Through comparing, exploring and self-explaining combined with the use of the interactive tool, the students managed to link different concepts together. The students were able to see and interpret the reason making the angle \( \theta \) and \( \sin \theta \) relate under certain conditions, thus leading to the standard trigonometric limit
\[ \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1. \]

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Trigonometry; standard trigonometric limit; conceptual understanding; procedural understanding; visualization; interactive technology environments; GeoGebra

1. Background

Although trigonometry is an essential field of mathematics, research in this field is quite limited (Kamber & Takaci, 2018; Moore et al., 2016; Weber, 2005, 2008). The importance of understanding trigonometry lies in the fact that it links algebraic, geometric, and graphical reasoning, which is essential for further understanding of pre-calculus and calculus (Weber, 2005). Research done in this field shows that students, as well as advanced adult learners, have difficulties due to limited or fragmented understanding in the subject (Chin, 2013; Moore & LaForest, 2014; Moore et al., 2016; Weber, 2008). Such deficits are leading to difficulties in future study of mathematics, and, e.g. engineering, physics and architecture (Moore & LaForest, 2014).

When working with right-angled triangles, students fail to interpret trigonometric operators (sine, cosine, and tangent) as ratios and to develop a function-based understanding of trigonometric operations (Weber, 2005, 2008). Students have difficulties linking different representations in trigonometry such as triangles, ratios and dynamically changing numerical relationships (Blackett & Tall, 1991; Kamber & Takaci, 2018; Weber, 2005, 2008).
Furthermore, confusion can arise from the students’ inability to reason about trigonometric functions in a circle context, thus failing to make connections between working with triangles and transferring this knowledge to circles (Chin, 2013; Moore & LaForest, 2014).

1.1. Visualization

In order to deal with the problem of understanding trigonometrical relationships, Blackett and Tall (1991) propose a digital approach to trigonometry that allows the students to ‘relate its dynamically changing state to the corresponding numerical concepts. It, therefore, has the potential of improving understanding’ (Blackett & Tall, 1991, p. 145). Blackett and Tall (1991) found that students who were taught trigonometry using interactive computer graphics showed better results than the control group who was taught trigonometry through conventional teaching methods. This was confirmed by research stating that visualization, using interactive technology environments such as graphing calculators or Geogebra, was recommended to improve students’ achievements in trigonometry (Abdul Rahman & Puteh, 2016; Choi-koh, 2003; Merrill et al., 2010; Naidoo & Govender, 2014; Prabowo et al., 2018). Research by Chen et al. (2015) indicates that integrating graphical learning materials into teaching enhances performance in pattern reasoning, and using digital learning materials can improve attitudes towards learning mathematics. Self-efficacy is defined as the personal conviction that one is capable to successfully organize and execute actions to produce educational outcomes (Bandura, 1977, p. 193; Zimmerman, 1995, p. 203).

2. Theoretical framework

Conventional trigonometry teaching has often been linked to procedural understanding based on memorizing rules and applying algorithms, instead of conceptual understanding based on establishing relationships between different mathematical concepts (Hirsch et al., 1991; Kamber & Takaci, 2018; Weber, 2005).

Procedural knowledge is defined as being the knowledge of procedures, algorithms and a sequence of steps executed to solve a familiar task correctly without necessarily involving reflection and deep understanding (Hiebert & Lefevre, 1986; Rittle-Johnson & Schneider, 2015; Silver, 1986). Conceptual knowledge is the knowledge of abstract concepts and general principles (Rittle-Johnson, 2017), and is a knowledge rich in relationships. Hiebert and Lefevre (1986) define conceptual knowledge as ‘a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information’ (Hiebert & Lefevre, 1986, pp. 3–4). A conceptual understanding involves the ability to switch between different representations of a concept (O’Callaghan, 1998). Conceptual knowledge can even play the role of validating critic, improving the student’s ability to judge the validity of an answer and to check whether it makes any sense (Brownell, 1947; Davis & McKnight, 1980; Hiebert & Lefevre, 1986). However, not all knowledge needs to be classified into either conceptual or procedural knowledge. Some knowledge is a mixture of both or even neither of them (Hiebert & Lefevre, 1986, p. 3).

According to Baroody et al. (2007), early research focused on oversimplifying procedural knowledge, thus classifying it as a shallow way of learning which includes memorizing algorithms (Baroody et al., 2007). On the other hand, a conceptual understanding was incorrectly described as a deep understanding while a procedural understanding is
described as a superficial understanding. In fact, research that problematized type and quality of knowledge and understanding shows that even a conceptual knowledge may be of a superficial quality and a procedural knowledge can be deep and sophisticated (Star, 2005, 2007; Star & Stylianides, 2013). As an example, flexibility in choosing an efficient strategy or algorithm, with critical judgment, is an indicator of deep procedural knowledge (Star, 2005, 2007). Conceptual knowledge cannot exist without knowing the tools and how to apply them (Kilpatrick et al., 2001). Hence, those two types of understanding are bidirectional. Both procedural knowledge and conceptual knowledge are connected by dynamic interplay; they complete and improve each other and are both needed to achieve a deep learning and a deep understanding (Baroody et al., 2007; Hiebert & Lefevre, 1986; Rittle-Johnson, 2017; Rittle-Johnson & Alibali, 1999; Rittle-Johnson & Schneider, 2015). Developing a conceptual understanding makes it even easier to remember procedures, rules and algorithms (Baroody et al., 2007; Carpenter, 1986; Hiebert & Lefevre, 1986; Silver, 1986) and to avoid the ‘tendency to use rules as reasons for action, without recognizing that using a rule is different from explaining why the rule works or why it is legitimate to use it in a particular case’ (Lampert, 1990, p. 56).

In order to achieve this co-transition between procedural and conceptual understanding, this study is based on a model with three pillars developed by Rittle-Johnson (2017):

- Comparing: when the student studies examples individually and reflect on them.
- Exploring before instruction: when the student is prompted to explore, discover patterns and pay attention to important information.
- Self-explaining: when the student generates explanation to make sense of the new information. Self-explaining has shown to be an effective way to help learners develop deeper understanding of the material they study (Ainsworth & Loizou, 2003; Rittle-Johnson, 2017).

Techniques such as comparing, explaining, and exploring promote procedural flexibility leading to more than one type of knowledge (Rittle-Johnson, 2017). Spontaneous self-explaining promotes new problem-solving approaches and deeper ways of thinking (Rittle-Johnson, 2006; Siegler, 2002), and leads to improvements in procedural learning, procedural transfer, and conceptual understanding (Carpenter et al., 1998; Chi, 2000; Chi et al., 1989). Comparing can improve all types of knowledge because it promotes perceptual learning (Rittle-Johnson & Star, 2011). Exploring problems and inventing solutions helps students to gain deeper conceptual and procedural knowledge of the topic (Rittle-Johnson, 2017; Schwartz et al., 2011). This study is based on observations where the students reflected out loud and were rarely interrupted by the teacher during this process. The idea is that students will explore, compare, and connect items leading to discovering relationships and learning through their own reflections and self-explaining. The model developed by Rittle-Johnson (2017) is found suitable for the purpose.

3. The purpose of the paper

The focus of this paper is to investigate high school students’ understanding of a trigonometric limit, i.e. $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$. The students are encouraged in a further step to explore
and understand the reason behind this limit according to the model developed by Rittle-Johnson (2017). This standard limit is very useful to know and understand in future mathematics learning, but also in, i.e. engineering, physics and architecture. In a study performed by Siyepu (2015), the author presents conceptual errors made by Chemical Engineering students showing that they did not grasp the concept of \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \). Thus the need to study this standard limit further.

This standard limit is first encountered by the Swedish students when they are dealing with the proof of the derivative of the function \( f(x) = \sin x \). At this stage, the students are asked to accept this fact with the motivation ‘you will get to understand it later in more advanced mathematics classes’. Sometimes the teacher shows a numerical explanation for the trigonometric limit using a table where the value of \( \theta \) decreases until the student sees the pattern converging towards 1. Regardless of which, the students did not see the reason behind this limit converging towards 1.

Aside from the numerical method mentioned above, there are several known methods to prove and compute the trigonometric limit \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \). The proof based on the Squeeze Theorem may be beautifully transparent and satisfactory, but is difficult to grasp for students fighting with their understanding of trigonometry. Applying L’Hôpital’s rule or using approximations based on Maclaurin series expansions are easy methods to compute the trigonometric limit \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \), without explicitly exploring the ‘why question’. In order to avoid a mere procedural understanding when dealing with trigonometry, the objective of this paper is to approach the task by combining conceptual understanding and procedural understanding. This objective is expected to be achieved through both visualization and the model inspired by Rittle-Johnson (2017).

The purpose of this paper is to follow the reasoning of high school students when asked to explain the standard limit defined above. This paper investigates if visualization and interactive technology environments can contribute to lifting the students’ understanding from a procedural understanding to a combination of conceptual and procedural understanding. The following research question is addressed:

- To what extent does visualization combined with comparing, self-explaining and exploring, support students’ understanding to graphically interpret and understand the trigonometric limit \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \)?

4. Method

This study is based on controlled observations, under arranged conditions. Each of the students met the researcher individually in a quiet room, where they could talk undisturbed. The room was equipped with a computer as well as a recording camera. The researcher explained the purpose of the study and the student was aware that he/she was being observed and that the conversation was video recorded. The camera could switch the focus between the paper the student was writing on and the screen of the computer with Geogebra. No faces were allowed to be recorded, no personal data was recorded. Each observation lasted approximately between 20 and 40 minutes.
4.1. Participants

McMillan and Schumacher (2010) stated that a non-random sampling method is the most efficient sampling method. Therefore, a voluntary response sampling was used in this study, as a form of a non-random sampling method. The chosen high school is a public high school situated in central Sweden, and the city’s largest high school with ca 1400 students. The participants were 13 Swedish upper-secondary high school students between 18 and 19 years of age, studying the technology program. The students had already taken the course Mathematics 4, i.e. a course that deals with trigonometry and with the concept of limits. After an introductory meeting with the researcher, the classroom teacher asked the class for volunteers for this study. All aspects for conducting ethically correct research were followed: the participants were informed about the study, the participants consented to be part of the study, the participants were informed about privacy, anonymity and confidentiality aspects related to the study, and that the collected data would be used only for the purpose of the study. All the ethical aspects were followed according to the Swedish principles of ethics in research (Vetenskapsrådet, 2017).

Each observation starts with the participating student being shown the detailed proof of how to get the derivative of $f(x) = \sin x$. The student is then told that one of the reasons for $f'(x) = \cos x$ is due to the well-known trigonometric limit $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$. At this point, this step is expected to be a repetition of what has already been taught by a teacher conducting a conventional teaching method. The observation is then divided into 4 phases:

**Phase 1**: A paper and a pen are handed to the student. The student is asked to interpret the formula $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

**Phase 2**: The student is handed a paper illustrating the unit circle. The same question as in **Phase 1** is asked. The student is expected to illustrate the reason behind $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ through graphical representations of the sine function using the unit circle.

**Phase 3**: The student is shown the interactive Geogebra application with the unit circle and a draggable point P on the unit circle periphery. The ray from the origin (0, 0) to the point P on the unit circle builds an angle $\theta$ from the positive x-axis. When the point P is dragged it is possible to observe how the angle $\theta$, the line segment in the unit circle representing $\sin \theta$, as well as the arc of the circle corresponding to the angle $\theta$ change accordingly. None of the parameters in the application is labelled. The student is invited to interpret what he/she sees and to associate it with the studied object.

**Phase 4**: The student is shown the same Geogebra application. All the parameters are labelled, and the application is dynamic, meaning that all the parameters change values when the student drags the point P (Figure 1). It is clear that when the angle $\theta$ is very small, all the values of the angle $\theta$, the line segment in the unit circle representing $\sin \theta$, as well as the arc of the circle corresponding to the angle $\theta$ become the same.

The entire observation is recorded, then transcribed and analyzed using a thematic analysis according to the model described by Braun and Clarke (2006). The process started with familiarizing with the data through transcribing and reading repeatedly, categorizing information, and identifying themes or patterns that generated codes and categories.
5. Results and discussion

The results and the discussion will be divided into before and after the visualization with the Geogebra application.

5.1. Before Geogebra – phases 1 and 2: inability to connect different representations

The students start with comparing and reflecting on the task according to the first pillar in the Rittle-Johnson (2017) model. The idea is that they will be able to switch between different representations of the concept of sine without any use of interactive technology environments. Those two phases witnessed clearly that the participants were unable to connect the different representations of sine. At this stage, it is difficult to decide if lack of knowledge is the reason. Later on, it became obvious that the students needed more information in order to connect all the concepts, compare them and draw conclusions. This was facilitated by the use of Geogebra and visualization of the concepts. The students showed no sign of conceptual knowledge. The lack of conceptual knowledge prevented them from going further in understanding, thus failing to interpret the reason making the limit \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \) go towards 1. This phase was associated with the students developing strong feelings of frustration due to the fact that the students were unable to ‘see’ patterns and associate concepts.

5.2. External factors revealed by self-explaining and exploring: time perspective, teacher impact

Linking information stored in the memory is a distinct sign of conceptual understanding. However, self-explaining and exploring before visualization showed that the students failed to remember knowledge stored in the memory. The time aspect relates to the property of failing to connect / to remember knowledge already stored in the memory. Students expressed that they forgot due to time elapsing (one year after their Mathematics 4 course).
They failed to remember formulas, they expressed feelings like ‘it has been a long while since we worked with this kind of mathematics’ or even ‘No, I have not seen this before’. Another external factor that was mentioned was the impact of the teacher. Students failed to remember because ‘the teacher never mentioned this before’ or even ‘I remember the formula, but the teacher never demonstrated the formula, but just skipped it’. This shows that in this case, both the textbook used in class, and the classroom teacher did exhibit a procedural attitude, leading to what can be qualified as procedural teaching. The student revealed that the classroom teacher’s explanation was based on procedural knowledge: giving formulas to memorize without any further explanation.

Many students blamed ‘the elapsing time’ for their forgetfulness. Time is an important factor in learning. It is true that one forgets with elapsing time and the question is if a conceptual understanding would contribute to a better understanding, hence keeping the information stored in the memory even after a longer period of time. This is in line with the core of conceptual understanding as defined by Hiebert and Lefevre (1986, p. 4). When it comes to teacher’s perspective, the students expressed a feeling that the teacher had been using a procedural teaching method, and the students were shown a formula without explaining the reason behind.

5.3. Deficit in the knowledge base

Self-explaining and exploring are known to be positive factors to improve students’ understanding for a task (Rittle-Johnson, 2017). In this case however, before visualization, self-explaining and exploring were tools that revealed some of the students’ misconceptions and common mistakes. Some basic knowledge that the students missed are listed here and analyzed afterwards. Notice, the expressions listed below are copied directly from what the students expressed orally or in writing.

- Suggesting that $\sin 90/90 = 1$, when trying to illustrate an example for $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$
- Mixing up sine and cosine in the unit circle
- Suggesting that $\sin 0 = 1$ and comparing with Geogebra and realizing that the reasoning is erroneous.
- Mixing up $\sin(1)$ and $\sin(90) = 1$
- Making their own formulas or accepting that $\sin 0/0 = 1$
- Suggesting that $1/0 = 1$
- Suggesting that $0/0 = 1$
- Suggesting that $0/0 = 0$
- Mixing up degrees and radians

The deficit in the knowledge base is also a logical explanation for the students’ lack of validating critic. For example, some students gave answers such as $\sin 90/90 = 1$ when trying to interpret the formula $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, incorrectly without the condition on the limits, implying that $1/90 = 1$. Such answers show that students missed details such as the condition put on the angle $\theta$ that should tend to zero. They also show that the students did not question if the answer is inappropriate or makes any sense (Hiebert & Lefevre, 1986, p. 13).

It is obvious that some students mixed up the two concepts sine and cosine. This can be due to their procedural knowledge where the formal language is a handicap, and there is a
lack of familiarity with symbols. This is even more obvious when some students could not relate sine of the angle to the y-coordinate value on the unit circle. This is again an example of failing to connect concepts and different representations, and failing to remember knowledge already stored in the memory (Hiebert & Lefevre, 1986, p. 4). The same can be stated about students making up their own formulas such as \( \sin 0 = 1 \), \( \frac{1}{0} = 1 \), \( 0/0 = 1 \) etc. The interesting part here is that when the students have memorized inaccurate information such as \( \sin 0 = 1 \), they realized that their reasoning is erroneous when they tried to verify it with Geogebra. This underlines the importance of a deeper understanding, which could be achieved through visualization (Abdul Rahman & Puteh, 2016; Blackett & Tall, 1991; Choi-koh, 2003; Merrill et al., 2010; Naidoo & Govender, 2014; Prabowo et al., 2018).

5.4. Context steered thinking

Exploring before instructions and trying to find strategies give the students opportunities to struggle and to figure out something that is not immediately apparent (Rittle-Johnson, 2017). In our case, many students were steered by the context while exploring. The context of limits was associated to derivative and prevented them from thinking about easy trigonometrical ratios. Instead they focused their attention on the concept of limits. One student wanted to associate the formula with the concept of integrals.

The context steered thinking blinded the students from seeing easy solutions. When students focused their attention on the concept of limits, they ended up failing in creating relationships between the existing knowledge and new information. Their attempts to find solutions related to derivative and limits show that their acquired knowledge is context bound with a clear tendency to compartmentalize knowledge (Hiebert & Lefevre, 1986, p. 18).

5.5. After Geogebra – phases 3 and 4: validating critic

The students tried to explore the task according to the second pillar in the Rittle-Johnson (2017) model. In this case, the students are paying attention to small details and important information using visualization through the interactive technology environment Geogebra. When equipped with conceptual knowledge a student can tell when a procedure is inappropriate, or when it is violating the conceptual principles (Hiebert & Lefevre, 1986, p. 12). The students questioned some results, and the conceptual knowledge was acting as a validating critic telling the students that some answers do not make any sense, prompting the students to re-evaluate the choice of procedure (Hiebert & Lefevre, 1986, p. 13).

The students were doing some self-explaining according to the third pillar in the Rittle-Johnson (2017) model, thus generating explanation to make sense of the new information and developing deeper understanding of the material. The students were using visualization through the interactive technology environment Geogebra. They were showing conceptual understanding when they managed to link together the two pieces of important information: the arc of the circle being the same as the angle, when one is dealing with the unit radians. According to Hiebert and Lefevre (1986, p. 4), this linking process between two pieces of information stored in the memory, or between an existing and one newly learned (in this case the arc of the circle and the angle of the triangle), displays a distinct sign that the students have acquired a conceptual understanding of the problem. This phase
is associated with self-efficacy and satisfactory feelings, due to the sense of achievement (Bandura, 1977; Chen et al., 2015; Zimmerman, 1995).

5.6. Associating representations and relating information through exploring and comparing

The students showed conceptual understanding when associating different representations of the sine function: a ratio in the right-angled triangle as well as the length of a certain line segment in a unit circle. Due to the Geogebra application, some of the students have even succeeded in demonstrating the formula for the arc length when $\theta$ is given in radians ($s = r \cdot \theta$). Some of the students were able to relate information and to associate concepts with stored information from a previous physics course. This is in line with the idea of acquiring conceptual understanding where previously unrelated items are suddenly seen as related (Hiebert & Lefevre, 1986, p. 4).

5.7. Reflecting on the details

Visualization combines the ‘three pillars’ of the model developed by Rittle-Johnson (2017) as well as connecting different representations which is the main core of showing a conceptual understanding. During this step, the students were prompted to explore and to find the patterns and how concepts relate to each other under certain conditions. Comparing, exploring, and self-explaining, are combined in this step. The students were able to switch between representations as well as to connect several concepts together, making his/her discourse lean towards a conceptual understanding.

The students were able to reflect and to associate the limit formula with the condition given on $\theta$. Due to Geogebra, the students found a pattern and recognized that the formula does not work if $\theta$ does not tend towards zero. While reflecting and self-explaining they were able to notice that some values are not the same if $\theta$ does not tend towards zero. This leads to the fact that the formula does not work unless some conditions are applied: ‘if you divide this value with this value, the ratio is not one’. To find a pattern and come to conclusions became much easier when the students, by means of visualization, were able to interpret the reasons and the conditions making this formula correct.

5.8. A combination of procedural and conceptual understanding

The results in this study could be summarized by Figure 2. In Figure 2, a discourse showing mere procedural understanding can develop to a discourse displaying a combination of procedural and conceptual understanding. This can be achieved with the use of digital tools in combination with the three pillars: ‘Comparing/Exploring/Self-explaining’ of the Rittle-Johnson (2017) model. A discourse showing mere conceptual understanding can also develop to a discourse displaying a combination of procedural and conceptual understanding following the same path.

The design in Figure 2 was applied to the discourse of the students who participated in this study. The students started with a discourse displaying a procedural understanding due to the formulas given by the classroom teacher to memorize without any further explanation. The use of digital tools for visualization was combined with the model with three
pillars developed by Rittle-Johnson (2017). Those two are on the same level because the students could go back and forth from the use of digital tools to comparing/exploring/self-explaining. The students even showed sign for validating critic as well as linking the different representations and concepts. The combination of these tools led the students towards a more conceptually based discourse.

An example illustrating one of the paths in Figure 2

Student: The first thing that I see is that when the angle approaches zero so both sine approaches zero and... aahhh of course the angle is approaching zero as well. Ahhh I get zero divided by zero. I do not know how I would… (A discourse displaying procedural understanding: the student is trying to execute an algorithm without succeeding)

Student: [...] This is how radians are constructed. Because, if you have radians, it becomes ahhhh... the arc of the circle has the same value as the angle. (A discourse displaying procedural understanding since the student is just repeating memorized rules without involving reflection: it is just the way it is; this is how they are constructed)

Researcher: Mmmmm very good

Student: I am still not able to associate this to... how this will approach... one. Aahhhhh it is actually
Researcher: What is happening when you drag point P making the angle approach zero as you are suggesting? What is happening with the arc of the circle?

Student: Ah the arc of the circle? Yes! This length of the line segment is… this length becomes of course, ahhaaaa this length is actually the sine for … (Use of digital tools for visualization, as well as Comparing + Exploring + Self-Explaining: the student is exploring through dragging the point P, then comparing the value of the segment line with the value of the arc of the circle and explaining what he/she discovered when dragging and comparing)

Researcher: Mmmm

Student: And if you make the angle smaller and smaller then, the arc of the circle becomes almost straight. Then the arc of the circle will be as equal as the line segment, they will be the same. Therefore, their ratio is one. Am I on the right track? (A discourse displaying a combination of procedural and conceptual understanding: the procedural understanding being the simple conclusion ‘they will be the same’ and the conceptual understanding being that the student related different pieces of information and came to the abstract conclusion that the ratio is 1)

Researcher: You are absolutely on the right track!

Student: Ahhhh this is exquisite!

Even when it comes to qualitative studies, quality can be assessed in terms of validity and reliability (Long & Johnson, 2000, p. 31). A number of factors may have affected reliability and validity in this study:

- The ambiguity when categories are defined or by the coding rules using the thematic analysis according to the model described by Braun and Clarke (2006). Will the reproducibility of the same study generate the same results?
- The relevance of the population. How significant is the choice of the students being in focus of this observation? Did emotional factors – such as stress, or being not at ease with the researcher – affect the students’ performance and their ability to succeed with the task already in phase 1?

Seeking higher levels of reliability and validity could be achieved through triangulation and the combination of several research methods to collect data for this specific study, which could be a future goal to achieve. Triangulation could be useful in future research in order to cross-validate data as well as to capture different dimensions and perspectives of the study. Since a non-random sampling was used (voluntary response), it is not possible to make conclusions about the entire population of high school students, but it was possible to develop an initial understanding of problems this population has in learning about the standard trigonometric limit, as well as how their learning could be improved.

6. Conclusion and future research

This study shows a positive correlation between visualization and the ability to connect different representations of trigonometric concepts, which enhances the conceptual understanding as shown in the results above. This enhancement of conceptual understanding
goes hand in hand with the students expressed satisfaction and self-efficacy. This is in line with research stating that self-efficacy can improve a positive attitude to mathematics, and that self-efficacy is enhanced by the use of digital tools (Chen et al., 2015, p. 1).

This study is yet to be generalized with data coming from a greater amount of participants. A future objective aims to do the same observation on a larger scale of population in order to validate what has been observed in this study, as well as to validate the choice of the categories. The author strongly believes in the positive impact of visualization in order to learn mathematics. Achieving a conceptual understanding through linking different concepts together can significantly be enhanced by the use of digital tools and different interactive technology environments.

**Disclosure statement**

No potential conflict of interest was reported by the author.

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