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Teaching algebraic thinking within early algebra – a literature review

Helena Eriksson
Dalarna University, Sweden; hei@du.se

There is a lack of overview regarding previous empirical studies within the early algebra research field. Consequently, the aim of this study is to propose one way to organise how algebraic thinking can be operationalised when teaching students five to twelve years old. The study is conducted as a literature review. The results show six categories of operationalising algebraic thinking with these young students. These categories can briefly be organised as three traditions: (1) arithmetic thinking tradition developing arithmetic thinking first, (2) developing arithmetic and algebra at the same time, or (3) algebraic thinking tradition developing algebraic thinking first. This method of organisation highlights one tradition of algebraic thinking where more research is needed - the tradition in which algebraic thinking is developed first. This tradition, as stated in the results, includes the category algebraic work.

Keywords: Early algebra, algebraic thinking, literature review, primary school students.

Introduction

In the early algebra research field, authors attempt to explain student opportunities to explore and discern mathematical relations, patterns and arithmetical structures through processes of noting, conjecturing, generalising, representing, justifying and communicating (Kieran et al., 2016). Early algebra is then manifested using symbols other than numbers only including geometrical figures, verbal and written language and gestures (Kaput, 2008; Kieran, 2004, 2018; Kieran et al., 2016). One problem within this research field is the age of the students and which level of the school system that is referred to as “early”. In the literature included in this review, early algebra can refer to the youngest students’ work on structures in mathematics, the introduction of algebra in secondary school or intermediate algebra, for example, as preparation for college-level mathematics (Katz, 2007). This broad focus on the level of schooling makes it difficult to navigate this field of research. Additionally, early algebra focusing only on the youngest students has been operationalised in different ways in different empirical studies (Blanton & Kaput, 2011; Kieran, 2004). Parts of this research concerns issues about how algebraic thinking should best be introduced to the youngest students. Hodgen, Oldenburg, and Ström skag (2018) argue, in a discussion on the last twenty years of developing research in mathematics education, that there is a need of an overview of this large number of different empirical studies regarding early algebra. Thus, it is difficult to navigate in this research field. The aim of this paper is to propose a method of categorising research regarding the teaching traditions of, or for, algebraic thinking in the age group five to twelve years old. The research question guiding the literature review is: According to which different traditions can algebraic thinking be operationalised within early algebra for students five to twelve years old?
Early algebra research and ways of operationalising algebraic thinking

This section briefly highlights different traditions of operationalising student algebraic thinking.

Research on algebraic thinking within early algebra concerns student actions related to ways of doing, thinking and talking about algebra (Hodgen, Oldenburg & Strömskag, 2019). This research also includes ways of operationalising algebra in teaching (Hodgen, Oldenburg & Strömskag, 2019). Kaput (2008) suggests two core aspects of algebraic thinking that may briefly be described as: (1) algebra as generalisations and expressions of the generalisations, and (2) algebra as guided actions on symbols within conventional symbol systems. Kaput (2008) further describes three strands of an embodiment of these core aspects: algebra as the study of structures, algebra as the study of functions, relations and statements, and algebra as the application of modelling languages. Kieran (2004) has proposed that algebraic thinking is connected to three interrelated activities for teaching school algebra broadly described as: (1) generational activities, for example forming equations, (2) transformational activities, for example rule-based operations, and (3) meta-level activities, as for example problem-solving in which algebra can be used as a tool. Further, Radford (2014) describes three ways of manifesting algebraic thinking as; (1) factual algebraic thinking, when students use their daily life language, (2) contextual algebraic thinking, when the symbols and language the students use are related to the specific context or situation, and (3) symbolic algebraic thinking, when the students use formal algebraic symbols. Davydov (2008) provides a fourth way of describing algebraic thinking related to the youngest students that is theoretically grounded on the idea that algebraic thinking develops if students can work with, and reflect on, arithmetical generalisations and that the youngest students are able to carry out such generalisations. Davydov (2008) argues that the young students should be introduced to algebraic work from the very beginning of their schooling and that students need to jointly take part in the work of identifying mathematical problems, choosing tools to work with, developing models to reflect on solutions and mathematical concepts, and lastly reflecting on whether the models developed are general and will work when solving other types of mathematical problems. The suggested tools when constructing these models include most algebraic symbols and geometrical figures (Schmittau, 2003).

Concerning the youngest students in the school system, van Oers (2001) describes three different traditions of mathematics teaching: (1) arithmetic thinking first – an arithmetic tradition in which teaching focuses on operations with numerical examples, (2) arithmetic and algebra at the same time – a problem-solving tradition in which teaching focuses on arithmetic and algebra as methods for solving tasks, or (3) algebraic thinking first – a tradition of algebraic thinking in which teaching challenges the students to identify mathematical problems and focus on what tools to use when solving these problems. In this third tradition van Oers (2001) suggest algebra to be used as a tool when teaching the youngest students.

Methods

This systematic literature review was conducted using the keywords; early algebra and algebraic thinking in the Education Resource Information Centre (ERIC) database. Early algebra was searched on 12 March 2018 and yielded 206 articles, algebraic thinking was searched on 26 October 2018, and yielded 331 articles. Fifty-one articles were identified in both searches. One observation due to the
date of the search is that any articles of a later date can be deductively organised into the categories described below.

While reading the titles and the abstracts of the 486 articles, a process of elimination was conducted in two steps. In a first reading, 274 articles were taken for further analyses including studies of students five to twelve years old. In a second reading, still based on the titles and the abstracts, 147 articles focusing on operationalising algebraic thinking in teaching were selected for further analyses. Articles concerning teachers or teacher students and articles about, for example, students with less ability in mathematics not focusing on teaching were omitted. They were omitted because these studies focused on how the participants understood algebra not on how to operationalise algebraic thinking. In total 147 articles were included in the extended analyses.

The next step in the analyses was to identify the descriptions of student opportunities to think algebraically. This was achieved by constructing thematic categories regarding operationalising algebraic thinking in early algebra according to the analysis question; Who is doing what with what tools and with what aim? (Eriksson & Eriksson, 2021). Here, the categories were inductively identified related to in which way the teaching of algebraic thinking was described in the studies. And finally, the categories found in this step of the analyses were interpreted and grouped into more overall traditions inspired by van Oers’ (2001) suggestions concerning different traditions for mathematics teaching; (1) as an arithmetic thinking tradition or arithmetic first (2) as a tradition of arithmetic and algebra at the same time or (3) as an algebraic thinking tradition or algebra first.

Results

The results of the literature review are presented in Table 1. This table includes a presentation of the six categories regarding operationalising algebraic thinking within early algebra.

Table 1:
The traditions, categories, and teaching examples given in the articles

<table>
<thead>
<tr>
<th>Tradition</th>
<th>Category</th>
<th>Examples of focus in teaching</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Arithmetic thinking tradition (arithmetic first)</td>
<td>1.a) Algebraized elementary mathematics</td>
<td>47-18+18=47</td>
</tr>
<tr>
<td></td>
<td>1.b) Pre-algebra</td>
<td>Numerical answers to unknowns</td>
</tr>
<tr>
<td>2. Algebra and arithmetic at the same time</td>
<td>2.a) Early algebraization</td>
<td>Operating with unknowns, equalities</td>
</tr>
<tr>
<td></td>
<td>2.b) Arithmetico-algebraic thinking</td>
<td>Relationships between different tools and notations</td>
</tr>
<tr>
<td></td>
<td>2.c) Emergent algebraic thinking</td>
<td>Geometrical patterns</td>
</tr>
<tr>
<td>3. Algebraic thinking tradition (algebra first)</td>
<td>3.a) Algebraic work</td>
<td>Structures between concepts jointly reflected using algebra, geometrical figures and language</td>
</tr>
</tbody>
</table>

1. Arithmetic thinking tradition

Category 1.a) is termed algebraized elementary mathematics and is related to the arithmetic thinking tradition. In this category, algebraic thinking is built on details manifested as arithmetic (Britt & Irwin, 2008; Lins & Kaput, 2004). Algebraic thinking could be developed by making structures visible using arithmetical examples. As one example, a statement such as 47+18-18=47 visualises
that whatever is added and then subtracted entails that the original number does not change (Lins & Kaput, 2004). To summarize this category: algebraic thinking is related to the idea that algebra is about generalisations that can, or need to be, developed from arithmetical examples (Britt & Irwin, 2008; Lins & Kaput, 2004). Thus, algebra is introduced after the students have developed their arithmetic abilities.

The category 1.b) includes studies presented as pre-algebra. These studies describe teaching that aims to prepare for algebraic teaching grounded in arithmetic (e.g., Carraher & Schliemann, 2007). Arithmetic is thus seen as operations with numbers, separated from algebra that is about generalisations. The teaching of pre-algebra in these studies is proposed to be positioned after arithmetic but before teaching algebra. The students are, for example, supposed to work with (1) counting, grouping and sorting artefacts, (2) numbers and numbers of objects, (3) comparisons of quantities and values, (4) organisation of sequences and (5) sums, differences, quotas, and products of quantities and values. To summarise this category: the focus is on numerical values, thus the arithmetic aspect of mathematics.

2. Algebra and arithmetic at the same time

Category 2.a), related to the algebra and arithmetic at the same time tradition and is termed early algebraization (Blanton & Kaput, 2011; Kieran, 2004). Early algebraization is based on the idea that arithmetic can be more than counting and by-heart knowledge (Blanton & Kaput, 2011; Kieran, 2004). The teaching described often focuses on student opportunities to analyse relationships between quantities, identify structures, generalise, solve problems, model, argue, prove and predict (Blanton & Kaput, 2011; Kieran, 2004). The studies representing this category state that the differences between arithmetic and algebra are not completely distinct, but the differences can be presented according to what is specific for algebra, thus: a focus on relationships, not counting numerically, a focus on operations and their inverses, a focus on the process of problem-solving, not only the answer, a focus on symbols such as numbers and letters and a focus on the meaning of the equals sign (Kieran, 2004). To summarise this category: algebra is used to analyse arithmetical relationships beyond numerical answers using numerical symbols.

In the category 2.b), teaching is focused on an arithmetico-algebraic way of thinking. The teaching within these studies describes, for example, a modelling process focusing on both arithmetic and algebra (Hitt et al., 2016; Pittalis, 2018). This type of teaching is categorised by student actions related to arithmetic, visible arithmetic processes, their transformations to algebra and their inverses. Students are supposed to develop arithmetic and algebra at the same time, consequently teaching focuses on relationships between different notations, tools and student actions (Hitt et al., 2016). To summarise this category: the studies identify points of contact between arithmetic and algebra instead of describing differences. Category 2.c) is termed emergent algebraic thinking and is suggested by, for example, Radford (2000, 2014) and Zazkis and Liljedahl (2002). Research interest concerns teaching focused on student ability to generalise and to then symbolise generalisations, thus students are allowed to express generalisations verbally using gestures and symbols without any requirement for students to note generalisations in a purely correct algebraic manner. Emergent algebraic thinking can thus be developed without using common mathematical nomenclature. The studies included are based on student opportunities to work algebraically by, for example, representing solutions to mathematical
problems verbally, by using written language, drawings, symbols, models and gestures. These actions form the observable data used to analyse the algebraic thinking developed in teaching (Roth & Radford, 2011). Problems are often manifested by geometrical pattern development. To summarize this category: emergent algebraic thinking focuses on different symbols, verbal language and gestures to operationalise algebraic thinking.

3. Algebraic thinking tradition
The final category 3.a), and the only category that is related to the algebraic thinking tradition, is termed algebraic work in which researchers often refer to the El’konin-Davydov Curriculum (Kozulin, 2003; Schmittau, 2003; Sophian, 2002; Venenciano & Dougherty, 2014). Here, teaching the youngest students begins with measuring and comparing quantities related to lengths, volumes, areas, weights and numerical values. These quantities are often noted using algebraic symbols and geometrical length segments. This type of teaching is operationalised as collective analyses of mathematical concepts, their derivation from measurement and their representation by schematic models. Relationships between quantities are identified by the students in collective problem situations. For example, a whole class may collaborate together with a teacher to develop models that visualise how a relationship noted as \( A = B + C \) can also be represented as \( C = A - B \) or \( B = A - B \). In order to be able to discuss such statements, the students and the teacher often use length segment models and algebraic symbols as in Figure 1.

![Figure 1: The relationship \( A = B + C \) as it is presented in Davydov (2008)](image)

Here, Algebra is used as a tool to discuss general structures of mathematical concepts (van Oers, 2001). The relationships depicted in Figure 1 can be used by students as a means for reflection on the essence of mathematics as scientific knowledge of quantity and relationships. In a next step students can compare quantities that are almost the same, quantities that do not differ significantly and thus need to be measured to be compared. Students may also compare quantities of lengths to quantities of weights to discuss if they are possible to compare. A third task may be to compare quantities of, for example, volume in containers of different shapes. Students then must identify that an intermediary unit is necessary in order to compare the different quantities. Such tasks can be designed without using numerical examples. The sets of progressively more difficult problems are not organised with different content but as problems in which previous solution methods are inadequate but give guidance. Students are supposed to identify the need for new methods, tools and conceptual knowledge. Increasingly difficult and complex problems are designed for the students to solve (e.g., Schmittau, 2003). It is proposed that this algebraic way of teaching is introduced to students from about five years old. The studies categorised as algebraic work under the algebraic thinking tradition often focus on student agency, their opportunities to initiate and take part in discussions on
mathematical content and reflect on structures and relations. Summarising this category, the students begin with algebra before arithmetic in joint activities.

Discussion

As the results above describe, the research field concerning early algebra is multifaceted in how algebraic thinking can be operationalised in teaching. Researchers in the literature review argue that it is not easy to obtain an overview of what algebra is and what arithmetic is when describing teaching in the different types of teaching traditions. This is managed here by categorising different teaching according to how algebraic thinking is operationalised. However, the same researchers argue that it is important to grasp differences and similarities between these two contents in order to better understand what is in focus and what is possible for the students to distinguish (Radford, 2000; Kieran, 2004, 2018). In this overview, these categories have been organised as: beginning with arithmetic and then introducing algebra, working with arithmetic and algebra at the same time and beginning with algebra to develop algebraic thinking as well as arithmetic thinking. Some concluding remarks are given in relation to the first and the third tradition.

In the arithmetic thinking tradition, arithmetic abilities are supposed to be developed before the students are expected to develop algebraic thinking. Within this tradition, algebra is usually introduced in middle school when the students have worked with arithmetic for a while and supposedly have developed basic arithmetic abilities. Even though these studies are included in the research field of early algebra, it is thus not the youngest students who are referred to in the research literature. This teaching tradition empowers students to communicate generalisations by using numerical examples as in, for example, pre-algebra (Carraher & Schliemann, 2007) and algebraized elementary mathematics (Lins & Kaput, 2004). According to Kaput’s (2008) description of algebraic thinking, this teaching can be understood in relation to generalised arithmetic, but less in relation to syntactical, guided manipulations of symbols. Based on Kieran (2016) and Kieran et al. (2018), this kind of teaching tradition can be understood as generalisation activities despite the symbols used being numerical. However, referring to Radford’s (2014) descriptions regarding manifestations of algebraic thinking, it is doubtful whether this is to be considered as algebraic thinking when the content is related to numerical values only.

In contrast to the arithmetic thinking tradition, the algebraic thinking tradition emphasis that algebraic thinking needs to be developed first in order to develop arithmetic thinking. Here algebraic thinking is operationalised in the earliest grades in elementary and primary school. Algebraic structures and relationships are worked with as a foundation for arithmetical work (Davydov, 2008; Schmittau, 2003; Sophian, 2002). The development of mathematical abilities using algebraic symbols and line segments is suggested as a means in a collective, problem-solving activity (Kozulin, 2003; Schmittau, 2003; Sophian, 2002; Venenciano & Dougherty, 2014). Teaching should focus on relationships between mathematical concepts and structures within arithmetic such as relationships between quantities (Schmittau, 2003). One important difference between this algebraic thinking tradition and the arithmetic thinking tradition is the idea that theoretical knowledge is developed by ascending from the abstract to the concrete (Davydov, 2008). In order to enable this process among the youngest students as well, algebraic symbols and algebraic ways of thinking are essential in the algebraic tradition. Comparing this way of operationalising algebraic thinking to Kaput’s (2008) three strands, this can be seen as: a) algebra as the study of structure and systems abstracted from computation, b)
algebra as the study of relationships and structures and c) algebra as collective modelling. Referring to Kieran (2004), the algebra first tradition is focusing generalisation work in which students and a teacher jointly construct equations and reflect on structures as they identify problems and explore mathematical tools.

The result of this literature review indicates that there is a lack of studies within the third tradition - starting with algebra. This gap is also identified by Coles (2021) in a discussion in Educational Studies in Mathematics, in which he states that "[t]he manner in which symbols arise from activity within Davydov’s work potentially offers huge advantages in multi-lingual classrooms, and I would see this as a rich area of future research” (p. 475). Consequently, this review confirms Coles’ (2021) argument that more studies in this field are necessary, specifically studies that could expand our knowledge of the algebraic thinking tradition.

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References


