FIVE YEAR OLDS IN BETWEEN SHARING AND DIVISION

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Abstract

Sharing and division are two concepts that have overlapping properties, and both are connected to the interpretation of fairness. In the present study, we study preschool children's work with a case where eight biscuits were shared between soft toys. The focus is onthe different arguments that the children express. The results show that children use both ethical arguments and mathematical arguments in their solutions. Some of the arguments can be categorised as 'Fair sharing related to number of pieces only' or 'Fair sharing employing ad hoc attempts at equal size'. The arguments that were coded as sharing not associated with mathematical sense of fairness were either classified as ethical reasoning or play. In the discussion, we raise the need of the combination of ethical reasoning and mathematical arguments if we want to create situations for children to develop critical thinking.

Keywords: division; ethical reasoning; mathematical arguments; preschool; sharing

Introduction

Division is one of the key concepts in mathematics and previous research has concluded that young children's understanding most often is a result of their experiences of sharing (e.g.Davis &Pitkethly, 1990; Desforges&Desforges, 1980; Squire & Bryant, 2002a, 2002b). At the same time, studies signal that every day experiences can be an obstacle for the understanding of division as sharing in equal parts (Smith et al., 2013; Wong & Nunes, 2014). Although it is easy to assume that division is a higher form of sharing, a fair share is not always the same thing as division(Hamamoucheet al., 2020;Hestner& Sumpter, 2018): it is about how resources should or could be shared (Chernyak&Sobel, 2016; Hestner& Sumpter, 2018; Smith et al., 2013). Looking at research in sustainability, sharing is one of the key questions (Latour, 2018; Pelletier, 2010), especiallysince we, according to Agenda21, need to be abstemious with resources (Zwarthoed, 2015). Sharing resources is also about the size of the parts: in the Brundtland report, it was concluded that resources need to be distributed more evenly to achieve sustainability (World Commission on Environment and Development, 1987). Also, inAgenda 2030,it is concluded that the earth's resources need to be better protected if we want to preserve oceans and save ecosystems (UN, 2015). This protection includesto make sure that no one is poor or hungry. Hence, there is an ecological dimension to the concept 'a fair share'when it comes to the earth's resources. The understanding of how to share resources in a sustainable way can resultin a distribution where someone gets less as the need is greater elsewhere, often low-income countries, ecologically disadvantaged regions, and poor households(UN, 2015). As such, the use of mathematics to decide what is a fair share is part of a larger social practice with ethical dimensions (e.g. Ernest, 2020).

Studies show that children's reasoning about sharing is affected by social and cultural contexts (Carson &Banuazizi, 2008; Huntsman, 1984; Sigelman&Waitzman, 1991; Wong & Nunes, 2014), also with respect to values (Chernyak& Sobel, 2016; Hestner& Sumpter, 2018).Preschool children often learn about sharing in preschool, as well from home and from friends (Borg, 2017). Given that 'fair' is not an unequivocal concept, values are therefore an important topic for teaching sharing, independent if the aim is to discuss values or to talk about division. It is therefore vital to create situations where children can explore, discuss, and reach conclusions, given it is through activities like these they will develop competencies and critical thinking (e.g.Öhman&Öhman, 2013), including ethical reasoning (Samuelsson, 2020). These competencies are crucial for sustainable development (Hedefalk et al., 2014; Jensen &Schnack, 1997; UNESCO,2017),hence the relevance to study young children's reasoning about sharing and division.

Background

In this paper, the focus is children's arguments about sharing. We will distinguish between mathematical arguments and ethical arguments.

Mathematical arguments

The starting point is to see mathematics as a social activity, especially in preschool where play is a factor (Sumpter &Hedefalk, 2015), and where negotiation is a central part (Voigt 1994) so that “learners participate as they interact with one another” (Yackel& Hanna, 2003, p. 228). However, compared to research on argumentation (AA), we wish to do a more detailed analysis of mathematical properties (e.g. Eriksson & Sumpter, 2021). Therefore, we use Lithner’s (2008) notion of anchoring including the structure of reasoning, allowing us to distinguish between what is relevant (or not relevant), and central (or not central):

> when deciding if 9/15 or 2/3 is largest, the size of the numbers (9, 15, 2, 3) is a *surface* property that is insufficient to resolve the problem, while the quotient captures the *intrinsic* property” (Lithner, 2008, p.261).

The first step in the structure is Task Situation (TS). It can be supported by identifying arguments (Eriksson & Sumpter, 2021): what is the task about? This step and its arguments can be compared to, however not the same,Pólya’s(1945)first problem solving phase. The second step in the reasoning structure isStrategy Choice (SC). It can be supported by predictive arguments (Lithner, 2008): why will the strategy solve the task? The third step is the Strategy Implementation (SI) andverifying arguments (Lithner, 2008): why did the strategy solve the task? The fourth and last step is the Conclusion (C), which can be supported by evaluative arguments(Hedefalk& Sumpter, 2017; Sumpter &Hedefalk, 2018): how and in what way the conclusion is an answer to the initial question?Such arguments are used in research in AI and serve the purpose to persuade that something is right or wrong (Carenini& Moore, 2006). An overview of the different arguments is presented in Figure 1:

Figure 1. Overview of steps in mathematical reasoning and different arguments.

Argumentation

Task Situation	Identifying arguments
Strategy Choice	Predictive arguments
Strategy Implementation	Verifying arguments
Conclusion	Evaluative arguments

Here, the focus is on the four different arguments and the content of the arguments. According to Lithner (2008), arguments anchored in mathematical properties are about objects (e.g. a rectangle), transformations (e.g. measuring a length or dividinga geometrical shape in equal sized parts), and concepts (e.g. the concept of measure) where concepts can have several objects and transformations connected to them.

Ethical reasoning

As we see it, in any step of the reasoning structure other arguments than mathematics can be used (Eriksson et al., 2022; Sumpter &Hedefalk, forthcoming), since what becomes meaningful depends on the problem at hand (Tväråna,2018).We define ethical reasoning as a collective line of arguments that is produced when solving a task, but where the arguments are anchored in values (Eriksson et al., 2022; Sumpter &Hedefalk, forthcoming). This definition is similar to moral reasoning (Samuelsson &Lindström, 2020), and the same criterions are therefore used when deciding whether an ethical reasoning is sustainable or not by using the SIL method (Samuelsson, 2020): (1) coherence (S) meaning that an ethical argument is coherent when it does not contain logical flaws for the individual expressing it; (2) information (I) which implies that the argument is correct, and has relevant information and motivations that a listener is willing to accept; and, (3) vividness (L) which means that the child needs to understand another person's (or soft toy's) point of view.The method is designed to encourage different types of ethical reasoning when teaching sustainability, and according to Samuelsson and Lindström (2020), all three steps are needed for an argument to be classified asan ethical argument. In this study, we convert the method to an analytical tool allowing us to analyse if different arguments can be considered as ethical argumentation(e.g.Eriksson et al., 2022; Sumpter &Hedefalk, forthcoming).

Divisionand sharing

Several studies have concluded that young children understand sharing including sharing in equal parts (e.g.Chernyak et al., 2016, Chernyak et al., 2019; Smith et al., 2013). The main difference between division and sharing is that the latter can accept unequal shares whereas division requires equal partitioning(Correa et al., 1998). Sharing and division has an overlap regarding some of the fundamental properties, such as both transformations need to take into account the amount that should be shared (dividend) and some considerations about the numbers of recipients (divisor) (Correa et al., 1998; Muldoon et al, 2009; Squire & Bryant, 2002a; 2002b). However, as stated earlier, the results from the different transformations can differ substantially: division require equal parts whereas sharing can allow unequal sized partitioning. Some characteristics of different arguments used by children when exploring fair sharing is found in Watson's (1997) study. The different children (age 4/5 – 8/9 years old) expressed four different ways when trying to share a pancake between three dolls. The first category was sharing not associated with mathematical sense of fairness. The second category was children using the numbers of pieces as an argument,but did not pay attention to the size of the amount. Strictly speaking, this is not division, although theargument about the cardinality could be seen as a mathematical argument. Similar reasoning has been found in Sumpter &Hedefalk (forthcoming) when children tried to solve 8/3 with the result {2 ½; 2 ½; 3} where the focus was on the number of objects in each group (*n*=3) rather than the size. The third category in Watson's (1997) study was when the children tried to make some attempt to address equal size, however these attempts were considered ad hoc solutions. One example of such solution could be cutting the paper into small bits and then compare the piles of paper clippings (e.g.Eriksson et al., 2022). The last category from Watson's (1997) study was when the children performed sharingusing geometric principles of a circle. Here, we would like to expand this category to general geometric figures to allow other geometrical shapes than just a circle. The first issue to address is the partitioning of the area into equal sized parts, which includes some sort of measuring (a transformation). This is an added complexity compared to comparing two separate lengths (e.g. Nunes, et al., 1993). Here, measuring could involve measuring the area of a 2-dimensional objects (a rectangle) or, if the shape is reduced in dimension, a line segment with the same length as the rectangle meaning that the relevant, intrinsic property is conserved.

Sharing can also be connected to different values. One example is Stemn's (2017) description of students from Liberia and their solutions of sharing $45 between three children of different ages:

> One student said they decided the oldest child would receive $20, the middle child would receive $15, and the youngest would receive $10. This method of sharing money and other items is not uncommon in many African cultures (Stemn, 2017, p. 391).

Such solution, in three unequal sized amounts, is considered sharing but whether or not it is a fair share depends on the arguments and if the arguments are accepted. Studies has shown that when solving mathematical tasks that involve sharing resources, children/ teenagers can use both mathematical properties and ethical properties such as values (e.g.Chernyak& Sobel, 2016; Enright et al., 1984). Examples of argumentsare if a recipient is identified having a greater need, or have made a greater effort.

Aim and research question

The aim of this paper is to study some children's arguments when facing a sharing scenario. In this study, we are interested of how children reason when a resource becomes scarce and where there are different needs making the task complex with no obvious solution.The research questions posed are: (1) What different arguments are children usingwhen solvinga task about sharing?; and (2), How can these arguments be characterised using Watson's (1997) categories?

Methods

The data is collected from a preschoolin a mid-sized town,centrally locatedin Sweden. The children (*n*=3) haveworked with several cases where the task in focus was the last one in a series of six (see Sumpter and Hedefalk (forthcoming) for further information about all cases). After working with the task, the childrenwere encouraged tomake a drawing to reflect about the task,and its solution. All work was video recorded. The story was the following:

> Today, we have two soft toys meeting up and they have biscuits. How many do they have? [count the biscuits: 8]
>
> How should these eight biscuits be shared? [If not 8/2 =4, encourage equal parts]
>
> They got four each. But here comes X who also wants to be part of eating biscuits. X is hungry and sad [move soft toy to emphasise feelings]. How should one share the biscuits then?

Here, one soft toy had different needs (i.e.being sad and hungry), and to make it a bit more complex, this individual was introduced after a first sharing had taken place meaning that all resources has already been distributed. Also, the chosen numberscontributed to the complexity: the dividend 8 is fixed but the divisor changes from 2 to 3. The case allows different solutions, such as division, either as grouping or repeated subtraction where the answer could include remainders or use of fractions. Other solutions could involve ethical reasoning using different types of arguments connected to ethics. The child could turn toobligation and morality to argue who needs biscuits (e.g.Tväråna, 2018). The child can also use consequences as an argument for distribution of biscuits, or virtue ethics such as one need to feed the one who is hungry. Another ethical argument can focus on care such that everyone needs biscuits, independent if one is hungry or full (Sumpter &Hedefalk, forthcoming).

The preschool is multilingual, meaning that all children have another home language. The children worked together with their preschool teacher and one of the authors, the latter in charge of the video recording and thereby having a subordinate role in the active work. In the transcripts, all names have been changed using generic names – Anna, Maya, and Nova –and the teacher is called 'Teacher', and the researcher 'Researcher'. The study follows the ethical principles of the Swedish Research Council, meaning that the parents have signed a letter of consent, and the children were asked if they wanted to participate. The children were also informed that they could stop working at any point without having to give any explanation.

The data was transcribed verbatim, including actions according to the principles presented by Mergenthaler and Stinson (1992). Although the children worked together, the decision was to categorise their separate arguments. The reason why is to highlight the different arguments presented and negotiated, which differ compared to studies of collective mathematical reasoning where the reasoning is treated as a joint activity (e.g. Eriksson & Sumpter, 2021; Sumpter, 2016). The transcripts were structured using the structure from Lithner (2008), meaning identifying each step (TS, SC, SI, and C) and the different arguments (identifying, predictive, verifying, and evaluating). The arguments were then analysed using Lithner's (2008) notion of anchoring in mathematical properties, and then compared to Watson's (1997) four different categories. The four categories were slightly modifiedas discussed earlier to suit the design of the present study.

The second step of the analysis was to do a further analysis of the arguments to see if they could be categorised as ethical reasoning using the SIL-method (i.e. Samuelsson, 2020). The three conditions, coherence (S), information (I), and vividness (L), are needed for an argument to be considered an ethical reasoning. As a last step, we analysed arguments not fitting the criterion for mathematical argument or ethical reasoning using an inductive approach.

Results

First, there is an overview of the results from the different categories including the results from the SIL-analysis. Then, each category is presented separately including examples from the data.

Table 1. Overview of the different categories

Description of the categories		**Names**
Sharingnot associated with mathematical sense of fairness	Ethical arguments	Anna Maya
	Play arguments	Maya Nova
Fair sharing related to number of pieces only		Anna Nova Maya

Fair sharing employing ad hoc attempts at equal size	Anna Maya Nova
Fair sharing using geometric principles of shape	Anna Nova

As illustrated in Table 1, the children used different arguments while solving the task. The decision was to split the first category into two after some further analysis.

1. Sharing not associated with mathematical sense of fairness

The two sub-categories derived from the content, ethics or play. The first example is when the lonely, sad, and hungry soft toy came into the situation.

2:46	Anna	What about this dog?
2:47	Teacher	Do you want to hear about this dog?
2:48	Nova	Oh, yes.
2:52	Teacher	It is sad. It is sad like this. [The teacher bends the headdown and keeps the toy in her knee.]
2:53	All children	Oh, here please. Take all the biscuits. [The children appear to be concerned about the thirdsoft toy. They pet and cuddle it.]
3:03	Anna	He gets them all. He is so sad.

According to the analysis, Anna's argumentation is ethical when applying the SIL-method. The argument to give all the biscuits to the sad dog is coherent (i.e. in this situation, it is accepted that the sad soft toy needs the biscuits), informed (i.e. she is repeating the information providedby the teacher), and vivid (i.e. Anna is using the expressed emotions as part of her argument). The next example is the category of play arguments, many of them related to social norms of behaving when eating, such asacting as friends would do:

3:08	Teacher	What about the other dogs?

3:11	Maya	Thank you. They are friends.
3:15	Teacher	If they are friends, is it then ok to give all biscuits to one of the friends?
3:17	Nova	They can eat together.

Other examples of arguments coded as play are ‘eating up’, and ‘learning to eat’. They arenotconsidered as ethical reasoning as they are not coherent, informed or vivid in relation to the task of sharing, and they are not considered mathematical since their content has no mathematical properties.

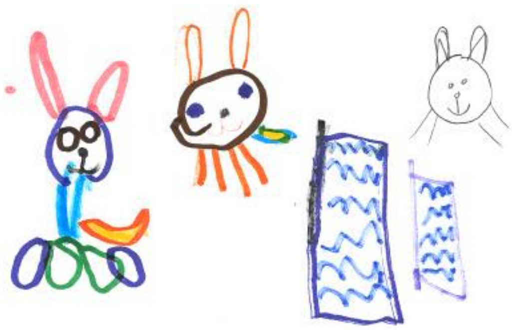
2. Fair sharing related to number of pieces only

The next category focus on the number of pieces, but not the size of the pieces. In the example below, Anna has shared six biscuits to the each of the soft toys (6/3 = 2, reminder 2). She is now focusing on the remaining two. Her strategy is to cut each of these biscuits into four pieces, and then each of these pieces is divided into smaller bits so there are 18 small bits of biscuits. She then distributes the bits and count them:

10:05	Anna	It is fair. All the dogs have eight biscuits. [Counts the wholebiscuits and the parts of the biscuits without any regard to the sizeof the pieces.]

In this example, Anna uses the cardinality of the number of pieces as an evaluative argument to why the conclusion is a solution to the task. In her drawing, the animal who is sad is marked out by not be drawn in colour(see Figure 2):

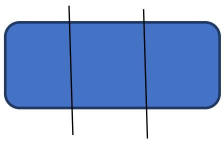
Figure 2. Anna’s drawing of the fair share.



She explains that it is hard to draw all the small bits of biscuits and therefore only two biscuits are part of the drawing.

3. Fair sharing employing ad hoc attempts at equal size

The next category, the children are using the size of the shares, however ad hoc attempts. This is illustrated with Novas's work: she gave pieces of biscuits to the soft toys, one at the time, until they had 3, 3, and 2 respectively. To make it fair/ equal she continued to cut the two biscuits allocated to the toy who had just two biscuits. The first biscuit she cut in three pieces and the other biscuit she cut in smaller and smaller parts.

14:20	Nova	Share all.
14:24		[Nova cuts the first biscuit in three pieces] 
14:24	Teacher	[Does it become] Is it more biscuit if you cut it into pieces?
14:43	Nova	Yes. [Cuts the other biscuit into small pieces.]
14:53	Teacher	Is it better to get such a biscuit? [The teacher points at the biscuits in pieces.]
15:05	Nova	No, they are all equal. Now it is fair. 

Nova starts by dividing a biscuit into three parts, indicating an understanding of partitioning in equal sizes. She expresses verifying argument during the implementation of the strategy, although they cannot be considered division (splitting an area in smaller bits do not change the size of the amount since the size of the area is conserved). Since there are more pieces (i.e. the cardinality increases), the strategy choice could be seen as an example of an argument that ‘Fair sharing related to number of pieces only’. However, in the next step of reasoning, Nova states that the piles created are the same size (‘they are all equal’), and therefore this episode is considered as an ad hoc attempt to achieve equal size. It is interesting to note, that in her drawing she presents a strategy where she is paying attention to the geometrical properties of splitting a rectangle in equal size parts (see Figure 3):

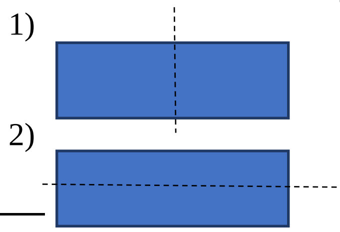
Figure 3: Nova’s reflection about the solution.

Nova explains that ”If we could share biscuits into nine pieces, it would be so much easier.” The data do not inform us why it is easier or how the nine pieces should be allocated, neither do we know why Nova think it is possible to share biscuits into nine pieces in theory but not in practice.

4. Fair sharing using geometric principles of a rectangle (biscuit)

The last category is illustrated with Anna’s attempt to share in equal parts[1].

5:12	Anna	I will cut across like this. [showing with a finger how to cut]



[1]The observant reader will note that the chosen episode takes place early in her reasoning, and the results presented in the category ‘Fair sharing related to number of pieces only’ is a second strategy choice.

		[The actions are showed twice, once for each biscuit]
5:27	Researcher	How many pieces do you get? You could ask your friends?
5:47	Anna	Eight.
5:52.	Teacher	[So] you are going to cut them in eight pieces. How are you going to share them?
5:56.	Anna	Like this. [She points at one soft toy at the time, however not in a specific order, and counts to 9] 1,2,3,4,5,6,7,8, 9. But then I have to cut one half more.

Anna's first argument (time code 5:12) is coded as a predictive argument supporting her strategy choice, to divide the rectangle into four equal sized parts. This could be described as an indication of reflective symmetry since the distance from each point to the symmetry line has equal distance. It could also be seen as measuring since the result is a structuring of the rectangle into composite units, where the size of the area is conserved. Anna appears to be content with the solution up to when she is asked about the allocations of resources, and she concludes she needs to transform one piece into two (as earlier discussed).

Discussion

Compared to Watson's (1997) categories, our results confirm her findings although with some adjustments. Here, the participating children expressed reasoning that had elements of several categories: for instance, Anna didn't fulfil her plan of using geometrical principles of the rectangle although she showed some initial understanding about conservation of area and partitioning into equal sized units which are two core aspects of measuring (Nunes et al., 1993). Nova, on the other hand, was able to draw a solution where she split the rectangle into nine pieces but her strategy choice did not include geometrical properties. It should be noted that the task in the present study differ from Watson (1997), which could explain some of the variations: here, the task was designed to allow different solutions, whereas the task used in Watson (1997) aimed to encourage equal partitioning. The four categories, however, were still valid. Here, there were several arguments classified as 'Fair sharing related to number of pieces only' or 'Fair sharing employing ad hoc attempts at equal size', a result that could be interpreted as apposite solutions with respect to age when comparing with Watson's (1997) results. Similar reasoning has been reported in previous studies (Eriksson et al., 2022; Sumpter &Hedefalk,

submitted).Our contribution is the expansion of the first category, 'Sharing not associated with mathematical sense of fairness', where weincorporated ethical reasoning and the SIL method, allowing an analysis of ethical arguments about sharing. This is an addition to the study of values and norms of sharing (e.g.Chernyak& Sobel, 2016; Enright et al., 1984; Stemn, 2017).Here, we show that there are two subcategories,play and ethical reasoning. Starting with play, given its central role in Swedish preschool (Sumpter &Hedefalk, 2015), we conclude that the category has a function here, sharing some aspects of the notion of vividness in ethical reasoning (e.g. Samuelsson, 2020). Continuing with ethical reasoning, given the global situation, with climate changes as a result from how resources have been used allocated and often without a sustainable perspective (e.g. UN, 2015; World Commission on Environment and Development, 1987; Zwarthoed, 2015), we think that this category is highly relevant to explore further. One suggestion for additional studies is to use Watson's (1997) design to see how young children changes in their arguments with age. Here, we do not see ethical reasoning as a less worthy type of reasoning. Rather, we suggest that this type of reasoning should be encouragedfollowing Ernest's (2020) idea of mathematics and ethics. Through such education, we argue, there is a chance to develop the competencies and critical thinking that is requested in Agenda 2030 (UN, 2015), and are crucial for sustainable development (Hedefalk et al., 2014; Jensen &Schnack, 1997).

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