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A PARABOLOID REFLECTOR APPROXIMATED BY SIMPLE SURFACES
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Abstract

A paraboloid of revolution is cut by planes parallel to its axis, dividing the surface into strips. These strips are approximated by bending certain strips made of flat reflecting material. Formulas for the shape of these strips are derived.
Introduction

A paraboloid of revolution is difficult to manufacture from e. g. coated aluminum sheets. A reflector close to a paraboloid of revolution can be made of concentric rings¹, but there are other ways to make a paraboloid-like surface, easy to produce.

Let the paraboloid $4fz = x^2 + y^2$ (here $f$ is its focal distance) be cut into strips by several planes $y = y_k$. Replace each strip by the surface obtained by letting a line segment glide, parallel to the yz-plane, with its end points on the boundary parabolas. This new strip can be flattened out on a plane.

This way to make a surface close to a paraboloid has some advantages. The shape of the focal region becomes a thin region with smaller area than the circular focal region obtained by using concentric strips. Also, material losses are kept small when cutting the strips out of (rectangular) sheets.

To the authors' knowledge, the formulas derived in this paper have not been deduced previously. A preliminary report was given at the 1st International Conference - Applied Optics and Solar Energy in Czechoslovakia².
Derivation of equations for the strips

Consider the surface $z = x^2 + y^2$ in Figure 1, given in an orthonormal coordinate system xyz (i.e. the three coordinate axes are pairwise perpendicular, and the unit of length is the same on the three axes). The surface is a paraboloid of revolution. Let two numbers $a$ and $b$, such that $a < b$, be given. Intersect the paraboloid by the planes $y = a$ and $y = b$. The intersections are parabolas. Denote them by $p_a$ and $p_b$ respectively. Let a real number $x$ be given. Intersect the paraboloid by the plane through the point $(x, 0, 0)$ that is perpendicular to the x-axis. The intersection is a third parabola. Denote it by $p_x$. Let $A$ and $B$ be the points where $p_x$ intersects $p_a$ and $p_b$ respectively. Let $V$ be the vertex of the parabola $p_a$ (see Figure 2), and let $C$ be a point on its positive half tangent at $A$ (then $C$ has a larger x-coordinate than $A$).
A problem

Find two functions $g$ and $h$ or, fuller

$$\xi = g(x), \quad \eta = h(x), \quad -\infty < x < \infty,$$

(1)

with these properties:

1. The functions $g$ and $h$ have continuous derivatives $g'$ and $h'$ with $g'(x) > 0$ on the entire $x$-axes.

2. Let $g$ be a given real number, and set $y' = (0, g)$. The functions $g$ and $h$ map the point $y$ onto the point $y'$ and the parabola $P_a$ onto a curve $C_a$ in the $\xi\eta$-plane, such that $C_a$ is symmetric with respect to the $\eta$-axis. Let $A'$ denote the image of $A$ under the mapping and, assuming $A' = (\xi', \eta')$, set $B' = (\xi', \eta' + |AB|)$; here $|AB|$ denotes the length of the line segment $AB$. Denote the locus of $B'$ by $C_b$.

3. The arc length $y'A'$ along the curve $C_a$ is equal to the arc length $yA$ along the parabola $P_a$.

4. Let $C'$ be a point on the positive halftangent of the curve $C_a$ at the point $A'$. The angles $BAC$ and $B'A'C'$ are congruent.

Comments on the problem.

(a) Consider the strip of the $\xi\eta$-plane bounded by the curves $C_a$ and $C_b$. By the listed properties of the functions $g$ and $h$ (in particular the properties 3 and 4), this strip can be bent to form a strip whose edges coincide with the parabolas $P_a$.
and \( P_b \). Further, any plane parallel to the \( yz \)-plane intersects the bent strip along a line segment of length

\[
|AB| = \sqrt{(b - a)^2 + [(b^2 + x^2) - (a^2 + x^2)]^2} \\
= (b - a) \sqrt{1 + (a + b)^2}
\]

(2)

Also, the bent strip approximates fairly well the strip of the paraboloid between the parabolas \( P_a \) and \( P_b \).

(b) We first tried to solve the problem numerically. The result indicated that the function \( \eta = h(x) \) probably is a polynomial of degree two. This induced us to try to find a solution in closed form. The following solution was obtained.

**Solution of the problem.**

\[
B = (x, b, x^2 + b^2) \\
C = (x+1, a, x^2 + a^2 + 2x) \\
A = (x, a, x^2 + a^2)
\]

**Figure 3.**

Consider Figure 3. It shows, among others, the points \( A, B, \) and \( C \) of Figure 2 and their coordinates. It is seen that the vectors \( \overrightarrow{0, 1, a+b} \) and \( \overrightarrow{1, 0, 2x} \) have the directions of the directed line segments \( \overline{AB} \) and \( \overline{AC} \) respectively. Let \( \Theta \) denote the measure (with \( 0 < \Theta < \pi \)) of the angle \( \overline{BAC} \). Use of the dot product gives

\[
\cos \Theta = \frac{(0, 1, a+b) \cdot (1, 0, 2x)}{\sqrt{1 + (a + b)^2} \sqrt{1 + 4x^2}} = \frac{cx(a+b)}{\sqrt{1 + 4x^2}}
\]

(3)

where

\[
c = \frac{2}{\sqrt{1 + (a + b)^2}}
\]

(4)

Assume that the functions \( g \) and \( h \) satisfy the problem.

Consider Figure 4. It shows, among others, the points \( V', A', B', \) and \( C' \) of Figure 2 and the coordinates of \( V' \) and \( A' \). The length of the arc \( V'A' \) is given by property 3, and
\( \Theta \) (as in (3)) is given by property 4. Let \( s(x) \) denote arc length, measured along the curve of Figure 4, normalized so that \( s(x) \) has the same sign as \( x \). Then (see the arc \( V'A' \) in Figure 4)

\[
\frac{ds}{dx} = \sqrt{1 + 4x^2}
\]

and (see (3) for \( \cos \Theta \))

\[
\frac{d\eta}{ds} = \cos \Theta = \frac{cx(a+b)}{\sqrt{1 + 4x^2}}
\]

It follows that

\[
\frac{d\eta}{dx} = \frac{d\eta}{ds}\frac{ds}{dx} = \frac{cx(a+b)}{\sqrt{1 + 4x^2}} \cdot \sqrt{1 + 4x^2} = cx(a+b)
\]

and \( h(x) = \eta \) gives

\[
\eta = h(x) = q + cx^2(a+b)/2
\]

Assume that \( \Theta \neq \pi/2 \). (Because \( \tan \pi/2 \) is not defined, we shall treat the case \( \Theta = \pi/2 \) separately.) Then

\[
\frac{d\xi}{d\eta} = \tan \Theta
\]

and Figure 5 gives

\[
\frac{d\xi}{dx} = \frac{d\xi}{d\eta}\frac{d\eta}{dx} = \tan \Theta \cdot cx(a+b) = \sqrt{1 + \left[ 4 - c^2(a+b)^2 \right] x^2} \]

\[
\sqrt{1 + \left[ 4 - c^2(a+b)^2 \right] x^2}
\]

\[
\frac{cx(a+b)}{\sqrt{1 + 4x^2}}
\]

\( \Theta \)

\[
\frac{cx(a+b)}{\sqrt{1 + 4x^2}}
\]
It is seen that the first and last members here are equal also when $\Theta = \pi/2$. The last equation can be simplified by noting that

$$\quad 4 - c^2(a+b)^2 = 4 - \frac{4(a+b)^2}{1 + (a+b)^2} = c^2$$

and, therefore,

$$\frac{d\xi}{dx} = \sqrt{1 + c^2x^2}$$

Hence, using $g(x) = \xi$ and $g(0) = 0$ (and a table of integrals) and denoting the natural logarithm by $\ln$:

$$\xi = g(x) = \frac{x}{2} \sqrt{1 + c^2x^2} + \frac{1}{2c} \ln \left( cx + \sqrt{1 + c^2x^2} \right) \quad (6)$$

The formulas (6) and (5) with the constant $c$ given by (4) solve the problem. It is readily checked that the functions (6) and (5) have the properties 1 to 4 listed in the context of (1).
Formulas for practical use

In the preceding section, the focal length \( f \) of the paraboloid was assumed to be 1/4. Introducing an arbitrary focal length \( f' \), the equation for the paraboloid becomes

\[
4f'z = x^2 + y^2
\]

For a strip that lies between the planes \( y = a \) and \( y = b \), the boundary curves are given by

\[
\xi_a = \xi_b = \frac{x}{2} \sqrt{1 + c^2 x^2} + \frac{1}{2c} \ln(cx + \sqrt{1 + c^2 x^2})
\]

(7)

\[
\eta_a = \frac{(a + b)c x^2}{8f}
\]

(8)

\[
\eta_b = \eta_a + w
\]

(9)

where the constant \( c \) is given by

\[
c = \frac{2}{\sqrt{16f^2 + (a + b)^2}}
\]

(10)

and the width \( w \) of the strip is given by

\[
w = |AB| = |A'B'| = \sqrt{(b-a)^2 + \left[\frac{(b^2 + x^2)}{4f} - \frac{(a^2 + x^2)}{4f}\right]^2}
\]

or, shorter,

\[
w = (b - a) \sqrt{1 + \left[\frac{(a + b)}{4f}\right]^2}
\]

(11)

Given \( f' \), \( a \), and \( b \), a computer program that gives \( \xi_a', \xi_b', \eta_a' \), and \( \eta_b \) for a given sequence of \( x \)-values is conveniently based on formulas (7) to (11).
References


SERC-REPORTS (ISSN 0284-1568)

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