Fair sharing and division – mathematical reasoning regarding integers and fractions in preschool and preschool class

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This paper identifies and discusses children's mathematical reasoning in preschool and preschool class when they work with a fair sharing case. The case came from a selection of cases designed to promote collective mathematical as well as ethical reasoning. Data comes from six children's work when sharing four paper biscuits between three soft toys, first when the children were five years old and then, a year later, when they were six years old. The results show that their reasoning, both when they were five and six, used mathematical and ethical arguments. In preschool class, the students were able to use each other's arguments in collective reasoning to identify, predict, and verify their reasoning. The students began to measure the fraction parts of a remainder but could not evaluate the conclusion with respect to what is aspects for division; equal numbers and equal size. The results also signal that teacher's input, of posing evaluating questions, appears to stimulate the reasoning.

Keywords: Fair sharing, division, fractions, collective mathematical reasoning.

Introduction

One of the key concepts in mathematics is division, which is a concept closely related to other concepts such as fractions, algebra, multiplication and so forth (e.g. Eriksson & Sumpter, 2021; Norton & Hackenberg, 2010), but also to sharing (e.g., Ching & Kong, 2022; Lamon, 2020; Steffe & Olive, 2010). The relationship between division and sharing is of interest since division builds on all recipients receiving equal amounts, whereas in sharing it is possible with solutions including unequal shares depending on, for example arguments of fairness (Watson, 1997). Previous studies have, for instance, focused on children moving from informal sharing to a more formal understanding of division. One example is Watson (1997) who suggests a framework where the first category is using no mathematical arguments when sharing to the last category covering solutions using geometrical principles of division. It builds on the idea of getting the children to understand that size is as important as cardinal quantities (e.g., Steffe & Olive, 2010). Watson's (1997) categories are useful from a mathematical point of view, but they do not allow other types of sharing. Also, it assumes that all children move from positive integers including zero ($\mathbb{Z}$) to rational numbers ($\mathbb{Q}$) whereas one could argue that this process can happen at different stages, and that mathematical solutions using indivisible items also are of interest. In addition, given that children learn through collective reasoning (Sumpter, 2016), since "learners participate as they interact with one another" (Yackel & Hanna 2003, p. 228), implies that teaching and learning is interpreted as a dynamic process aiming for meaning making behaviour (Sumpter & Hedefalk, 2018). Here, the aim is to study children's mathematical reasoning when they are working with a fair sharing problem when they are five years old and when they are six years old. The research questions posed are: (1) What different reasoning do the students express?; and, (2) What different arguments are part of the reasoning?

Background

Mathematical reasoning can be described in different ways. Following the idea that learning is a social process (van Oers, 2018; Yackel & Hanna, 2003), we explore children's collective mathematical reasoning. Collective mathematical reasoning is defined as the line of thought which is a result of collective joint work, work that can be organised as lines of arguments (Sumpter, 2016). Here, we differ to reasoning described as an individual process (e.g., Lithner, 2008). When identifying the different arguments of the reasoning, we organise the mathematical reasoning using the following structure: (1) a (sub)task is explored (TS); (2) a strategy choice is made (SC); (3) the strategy is implemented (SI); and, (4) a conclusion is suggested (C) (Lithner, 2008). For each step, different arguments can be expressed (Eriksson & Sumpter, 2021; Lithner, 2008; Sumpter & Hedefalk, 2018). We find identifying arguments connected to the first step, the task situation, predictive arguments are linked to the strategy choice intended to answer the question "Why will the strategy solve the task?", strategy implementations are supported by verifying arguments asking "Why did the strategy solve the task?", evaluative arguments answering the question "How does the conclusion answer the question for the (sub)task explored?". These different types of arguments can be anchored with mathematical properties in the reasoning process (Lithner, 2008). Mathematical properties are described as objects, transformations, and concepts. When working with sharing, it is also possible to use ethical reasoning when, for instance, allocating more resources to someone in greater need (Eriksson et al., 2024; Hedefalk et al., 2022; Sumpter & Hedefalk, 2023). Here, we are interested in properties related to fair sharing connected to division and related mathematical concepts.

Previous research has shown that young children are able to develop an informal understanding of division when working with a sharing problem (Ching & Kong, 2022; Watson, 1997). Some of the mathematical properties of division are more difficult to reason about, for instance, that the inverse relationship between the quotient and the divisor are more tricky compare to the equivalence principle. This means that working with different divisors is more difficult than working with the same divisor (Ching & Kong, 2022). Another aspect of division that is a core feature is the relationship between the number of parts and the size of each part in the divisor (Lamon, 2020; Watson, 1997). One solution of experiencing division is to see it as fractions, and to use unitizing proper fraction (e.g., 1/3) and thereby develop a fraction language (Steffe & Olive, 2010). Some researchers argue that before it is possible to develop schemes with rich connections for fractions, children must experience these equal parts of the same size as a standpoint for division as well as fraction (Norton & Hackenberg, 2010; Steffe & Olive, 2010). Such ways of thinking are in line with Watson's (1997) four categories: (1) sharing which appears to have no association with fairness in a mathematical sense; (2) fair sharing related to the number of pieces only; (3) fair sharing associated with ad hoc attempts at equal sizes; and, (4) fair sharing determined using geometric principles. The last two categories relate to the idea of seeing fractions as measures (Lamon, 2020).

Method

Data consists of children's work with a sequence of six cases, where the cases are presented in Sumpter and Hedefalk (2023). The work with all six cases was included in a project aimed to explore teaching about fair sharing and sustainability. Due to one of the aims in that project, exploring young students reasoning about fair sharing, a first round of data collection included six children (age five years old) working in pairs: (1) Noel (age 5y 8m) and Maya (age 4y and 9m); (2) Nova (5y and 2 m) and Ida (5y 1 m); and, (3) Adam (5y 6m) and Anna (5y 2m). Due to another aim in the project, to identify what reasoning students express in different ages, in a second round, a year later, the same three of these children partook, working together as a group (Adam, Nova, and Ida). The same preschool teacher was present at the two sets, and the same instruction was given on both occasions: asking questions to stimulate arguments, such as "What are your thinking?" and "Why do you think that?". All children are born in Sweden except for Noel who arrived in Sweden three months prior to the first recording, they all have another first language than Swedish. This choice of children was done due to the interest from the school organisation of developing mathematics education. Here, due to space limit, we present the children's work with the second case, Case 2, where four (paper) biscuits should be shared between three recipients (soft toys), see Figure 1.

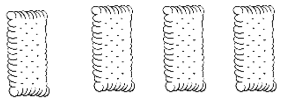


Figure 1: The biscuits and the soft toys in the case

In Case 2, the children faced the issue of the need for a fraction if dividing the biscuits equal to the recipients. Different soft toys were used in the different cases in order to prevent the students developing feelings for one specific soft toy. The cases were presented to the children that the three toys were visiting us and we have received biscuits from a factory (cf. Sumpter & Hedefalk, 2023). The children's parents have signed a letter of consent following the Swedish Research Council. All names are changed, and the children are informed that they at any time can drop out of the study, not having to explain why. Besides the preschool teacher, the first author of this paper was present. The first author videotaped and in the first step of the analyses transcribed these videos verbatim including actions (e.g., Mergenthaler & Stinson, 1992). The second step of the analyses, the transcripts were organised according to the structure for mathematical reasoning (Lithner, 2008): (a) task situation – TS; (b) strategy choice – SC; (c) strategy implementation – SI; and finally, (d) conclusion – C, see Table 1. The third step was to analyse the different arguments (identifying, predictive, verifying and evaluating arguments) with respect to the mathematical properties that was used in the argument.

Table 1: The analyses of the arguments in the different categories

Category	Mathematical reasoning
Sharing not associated with mathematical sense of fairness	TS – identifying argument SC – predictive argument SI – verifying argument C – evaluative argument
Fair sharing related to number of pieces only	TS, SC, SI, C
Fair sharing employing ad hoc attempts at equal size	TS, SC, SI, C
Fair sharing using geometric principles (size and numbers)	TS, SC, SI, C

The final step of the analysis was to categories the arguments using Watson's (1997) four categories: (1) sharing which appears to have no association with fairness in a mathematical sense; (2) fair sharing related to the number of pieces only; (3) fair sharing associated with ad hoc attempts at equal sizes; and, (4) fair sharing determined using some kind of geometric principles (numbers and pieces).

Results

The results show the mathematical reasoning expressed and the arguments used by the children and the teacher when the children are five and when they are six years old. In Table 2, we can see what categories of fair sharing that anchored the children's mathematical reasoning:

Table 2: The children's reasoning regarding fair share

Category	Mathematical reasoning	
	5 years old	6 years old
Sharing not associated with mathematical sense of fairness	Nova and Ida	
Fair sharing related to number of pieces only	Noel and Maya	Adam, Nova, and Ida
Fair sharing employing ad hoc attempts at equal size	Adam and Anna	Adam, Nova, and Ida
Fair sharing using geometric principles (size and numbers)		(Adam, Nova, and Ida)

As we can see in Table 2, there are some differences in the reasoning in the first round compared to the second one. The five-year-old children, used three different types of reasoning when sharing four biscuits between three recipients. Two of the children, Nova and Ida, made the strategy choice to group the biscuits in groups of two. The consequence when implementing this strategy was that one of the recipients didn´t get any biscuits. As a conclusion, they argued that this didn´t matter, with the argument "He [the dog] can eat grass", an argument categorised as having no mathematical properties connected to a mathematical sense of fairness. The next group of children, Noel and Maya, cut the fourth biscuit into small pieces, shared these pieces, and counted them using only the cardinal quantity. Also, they decided one animal was hungrier so this animal could get more biscuits. The third pair of the five-year-old, Adam and Anna, had no problem sharing the first three biscuits but then expressed different opinions of how to share the remaining biscuit. One child wanted to share the remaining biscuit in three parts whereas the other in four parts. The biscuit got cut in four parts, which was directly identified and formulated as a problem. "Oh no, 4 parts again". Their reasoning was categoriesed as ad hoc attempts at equal size.

When the children one year later, as six-year-old, an overall TS was identified at the beginning of the work when one child identified that "equally fair" was the aim for this case. This aim challenged these three students through the reasoning process. Then, three TS emerged in the collective work. The first TS included the SC to distribute the biscuits, one by one, to the soft toys, with the result that there was one biscuit left, (i.e., remainder 1). A second TS started when the teacher reminded the children that all biscuits should be distributed. The SC then was to divide the remaining biscuits in three parts, where they expressed both predicting arguments and evaluating argument during this process. However, there were no arguments using the mathematical property ‘size of the pieces’. This reasoning was followed up by a child holding a pair of scissors and carried out the actual dividing by cutting the remainder, moving the scissor back and forth on the biscuit before cutting, measuring where to cut, which resulted in three bits of different sizes. This transformation cannot be categorised as equal sizes and is therefore considered as employing ad hoc attempts at equal size. However, it is closer compared to a reasoning where there is no attention at all to the size of the pieces. Before the girl implemented the SC, she expressed predictive arguments when the teacher asked her how she was going to cut. The other two children agreed to the SC, and the girl cut one piece slightly smaller than ½, and two pieces slightly larger than ¼ (see Figure 2):

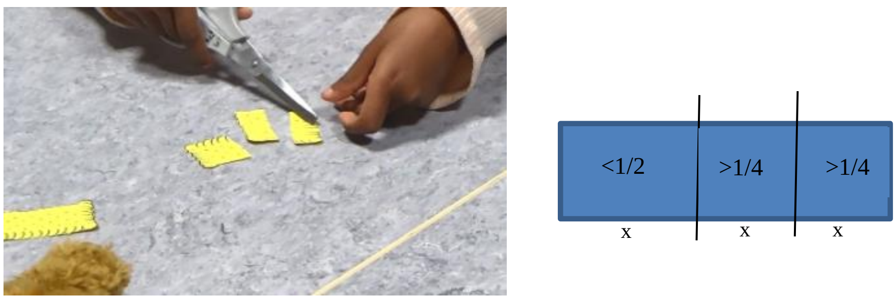

Figure 2: The pieces when the 6-years old have cut the fourth biscuit

The reasoning was made collectively and an evaluating argument in the Conclusion was expressed by Adam:

0307 Adam: No stop. Don´t cut anymore. It should be here [gives the pieces to the soft toysas in Figure 3].

Figure 3: The pieces when the 6-years old have distributed the fourth biscuit to the soft toys

The children delivered the three pieces to the recipients as if they were equal in size (Figure 3). The teacher challenged the students initiating a possible third TS as a result of asking for evaluating arguments to the Conclusion:

0324 Teacher: Do they have the same amount of biscuits?

0324 Adam: I knew you should ask that question. Look, 1, 2 and 1, 2 and 1, 2 [points to the two pieces of different sizes beside each toy].

0340 Teacher Is it the same amount of biscuits?

The children started to compare in different ways: Adam above points out the cardinality of the number of pieces (*n*=2). To the second question made the children compare the size of the pieces concluding that the length and the thickness of the three divided pieces was equal:

0407 Teacher: Have you got three pieces of the same size?

0411 Adam: [Places the two smallest biscuits parts next to each other]. They are at the same lengths.

0418 Teacher: Is this piece also the same size?

0419 Ida: [Gives the third piece of the remainder to Adam.]

0421 Adam: [Places this third piece, the one that is some larger than the other two, next to the other pieces, see Figure 4.] No, but it is at the same thickness.

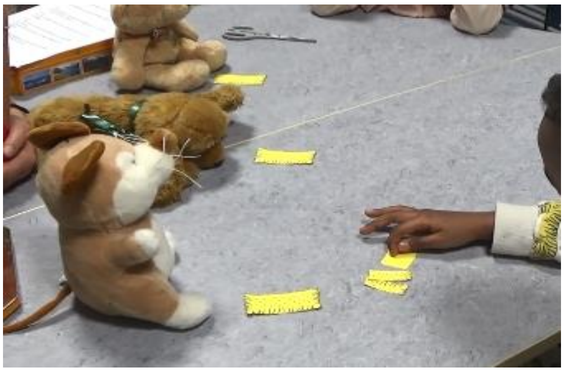

Figure 4: The comparison of the three small pieces of the remainder

0457 Teacher: What do you mean?
[None of the students answered this question.]

When the teacher asked for further arguments, the children had no suggestion of SC, SI or C, so therefore, the decision here to categorise this TS as a start of reasoning in the category fair sharing determined using some kind of geometric principles.

Discussion

Looking at the overall mathematical reasoning, the children moved from play and somewhat focus on cardinality and no explicit attention to the intrinsic property ‘size’ to more attention to cardinality and some attention to size. Similar transition is described by Watson (1997). But compared to Watson (1997), our results also show that the children started to express more different types of arguments in different stages of the reasoning, to each other and to the teacher. One example of reasoning as collective effort is the strategy choice and its implementation of the second task situation. The teacher asked the child what she was going to do, and she expressed predictive arguments. The two other children confirmed the strategy choice before it was implemented. This is an example of collective mathematical reasoning as a joint activity where actors contribute to solution (Sumpter, 2016). Our results and their difference to Watson’s (1997) results are, of course, related to the different design of the two studies. Hence, differences in the results are very likely to be linked to methodological aspects. Nevertheless, the results are of interest since one of the key problems in mathematics education is to get children/ students to express mathematical arguments (Sumpter, 2016) especially when teachers tend to take over the reasoning (Sumpter & Hedefalk, 2018). This is not the case here: the teacher was acting as a support for them to develop their reasoning, from the first task situation to the last one, where they explicitly made the decision not to pursue to a final conclusion. She helped them to pay attention to relevant, intrinsic mathematical properties of the components of the task (e.g., Lithner, 2008), and although they did not fully incorporate the last step of Watson’s (1997) categories, the children show signs that measuring, and size are vital aspects of division. If one of the key goals with mathematics education is to provide children with appropriate mathematical learning experiences so they get to develop the abilities we want them to do (e.g., van Oers, 2018), the teacher’s input is vital, independent how thoroughly and careful the cases have been designed (e.g., Sumpter & Hedefalk, 2023). We would like to suggest this topic as relevant for further research especially when taking the discussion ‘what is a key question’ (e.g., Sumpter & Hedefalk, 2018) to account.

Another important result is the explicit increase of using measuring as a transformation in their reasoning. In division and fraction, measuring is a key feature to create equal sized pieces (Norton & Hackenberg, 2010; Steffe & Olive, 2010). It functions as a transformation (e.g., Lithner, 2008), but one can also view fraction as a measure (Lamon, 2020). In Watson’s (1997) study, the older children were more likely to use measuring, either as ad hoc or based on relevant mathematical properties. Our studies confirm such conclusion (Eriksson et al., 2024; Hedefalk et al., 2022), including the present one. A plausible prediction is that when these children are seven years old, measuring will be part of their identifying arguments: how to make it “equally fair”. It also means that other solutions might be lost such as those supported by an ethical reasoning (Eriksson et al., 2024; Hedefalk et al., 2022; Sumpter & Hedefalk, 2023). We therefore conclude, that the move from integers to rational numbers will gain in mathematical properties relevant to fractions, but not necessarily helping children working with indivisible items. Here is a challenge for us, as teacher educators and researchers, to explore what is lost in this transition.

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