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A SUN CELL CORNET UNFOLDED

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Contents

1. An economic and technical problem

A sun cell is used to transform radiant energy into electric energy. A sun cell is an expensive device. We want to reduce costs by placing on a sun cell a cornet, manufactured by material with a good reflection property. (See Figure 1, where a sun cell, circular in shape, is situated at the bottom of a wide cornet.)

We want to give the opening of the cornet a square shape. The reason for this is a wish to be able to place many sun cells on a panel and collect all the sun energy flowing to the panel.

We also want to cut out the cornet from a planar sheet (of aluminum, specially prepared for reflection) and then bend the planar outcut into the wanted cornet shape.

What form should we give to the outcut?

This article has the purpose to give an answer to the stated question (and to discuss some related questions).

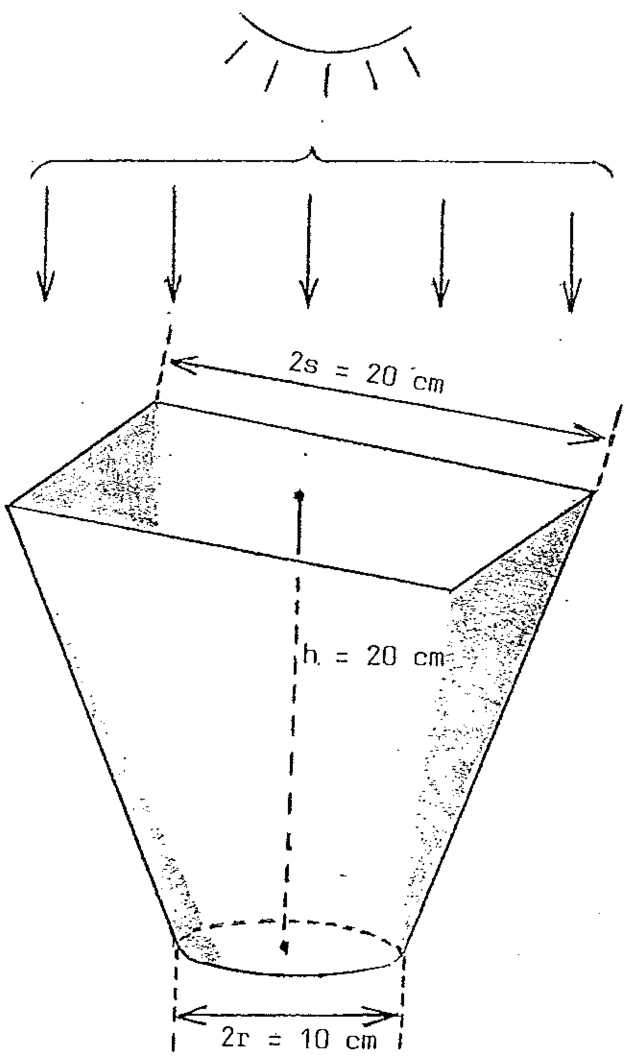


Figure 1. The sun cell cornet.

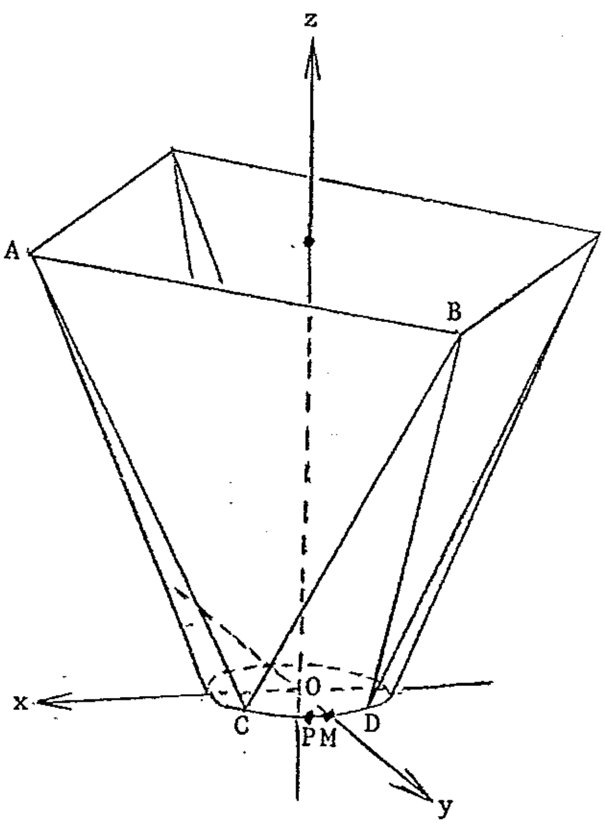


Figure 2. The coordinate system xyz.

2. A plan for solving the problem

Let three lengths s, r, h be given. Here s is half the side of the square opening, r is the radius of the circular bottom, and h is the height of the cornet. (The square and the circle are in parallel planes, and the height joins their centers and is perpendicular to their planes.) Figure 1 shows numerical values of s, r, h to be used in examples in the following. We let the unit of lenght be 1 centimeter; however, we usually do not write the unit in figures and formulas.

We introduce a coordinate system xyz as in Figure 2: The circle has the equations $x^2 + y^2 = r^2$, $z = 0$; the vertexes of the square are the points $(\pm s\sqrt{2}, 0, h)$ and $(0, \pm s\sqrt{2}, h)$. We denote by A, B, C, D, M, O the points $(s\sqrt{2}, 0, h)$, $(0, s\sqrt{2}, h)$, $(r/\sqrt{2}, r/\sqrt{2}, 0)$, $(-r/\sqrt{2}, r/\sqrt{2}, 0)$, $(0, r, 0)$ and the origin, respectively.

We let the cornet consist of four congruent planar parts and four congruent curved parts: One planar part has the isosceles triangle ABC as boundary; one curved part has the union of the line segments BC, BD and the circle arc CD as boundary. This curved part is a subset of an elliptic cone (a skew circular cone), having its vertex at the point B.

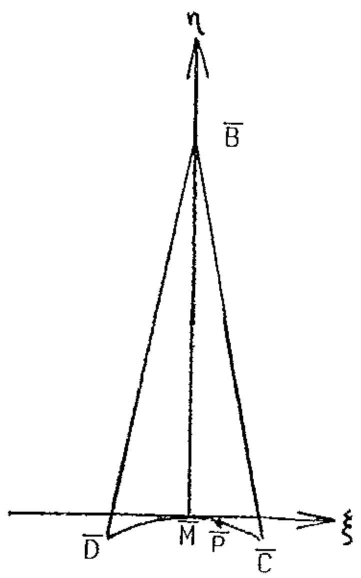


Figure 3. The part BCD of the cornet unfolded.

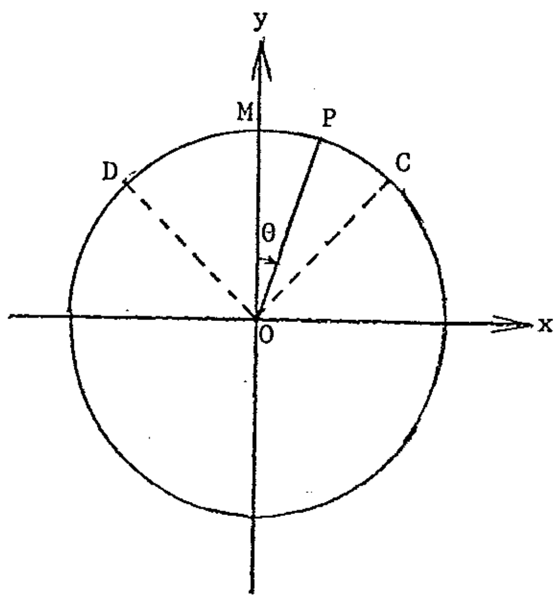


Figure 4. The circle (the sun cell) in the xy-plane.

Imagine that we cut the cornet along a line segment (for example, the line segment with the end points A and (r, 0, 0)). We can then unfold the cornet onto a region in a plane; we shall call this plane the $\xi\eta$-plane. The part BCD of the cornet is unfolded onto a region such as the region $\overline{B}\overline{C}\overline{D}$ in Figure 3; we denote the images (the new positions) of B, C, ... in the $\xi\eta$-plane by $\overline{B}$, $\overline{C}$, ... respectively. We place the axes so that $\overline{M}$ is the origin, that $\overline{B}$ is situated on the positive η-axis, and that $\overline{C}$ has a positive ξ-coordinate.

This cutting and unfolding solves, in a sense, the problem in Section 1. (Does there exist other solutions? The answer is yes, but the other solutions are not of interest in practice. They will be discussed in an appendix.)

There remains, however, some detail work.

Let a variable point P (see Figures 2 and 4) describe the circle arc MC, i.e., the set of points $P = (r \sin \theta, r \cos \theta, 0)$, $0^0 \leqslant \theta \leqslant 45^0$, with θ defined as in Figure 4; we measure angles in degrees. (The region $\overline{B}\overline{C}\overline{D}$ in Figure 3 is symmetric with respect to the η-axis; therefore we abstain from letting P describe the entire arc DMC.) Then the corresponding point $\overline{P}$ describes a curve $\overline{M}\overline{C}$ in the $\xi\eta$-plane (see Figure 3).

It does not seem possible to deduce an equation (in the sense of analytic geometry) for the curve $\overline{M}\overline{C}$. We therefore use a computer to find approximations of a (large) number of points on the curve.

To this end, let a positive integer n be given (in an example in Section 3, we set n = 9), and divide the circle arc MC into n congruent arcs, by points that we denote

(a) $$P_0 = M,\ P_1,\ P_2,\ \ldots,\ P_{n-1},\ P_n = C.$$

(It is left to the reader to draw a variant of Figure 4, on paper or in his imagination.) Each circle arc $P_{i-1} P_i$ $(1 \leqslant i \leqslant n)$ has the center angle $45^0/n$. We denote the common length of their chords by c:

(1) $$c = |P_{i-1} P_i| = 2r \cdot \sin (22.5^0/n),\ 1 \leqslant i \leqslant n.$$

To the points (a) we shall determine (step by step) images in the $\xi\eta$-plane, denoted

(b) $$\overline{\overline{P}}_0 = \overline{M},\ \overline{\overline{P}}_1,\ \overline{\overline{P}}_2,\ \ldots,\ \overline{\overline{P}}_{n-1},\ \overline{\overline{P}}_n \approx \overline{C}.$$

(Draw a variant of Figure 3. We use the sign $\approx$ in the meaning "is approximately equal to".) We set $\overline{\overline{P}}_i = (\xi_i, \eta_i)$, $0 \leqslant i \leqslant n$.

Assume that some points, up to the point $\overline{P}_{i-1}$, in the beginning of the sequence (b) have already been found. We then determine $\overline{P}_i$ by the three conditions:

$$(c) \qquad |\overline{P}_i\overline{P}_{i-1}| = c, \quad |\overline{P}_i\overline{B}| = |P_iB|, \quad \xi_i > \xi_{i-1},$$

We do this for $1 \leqslant i \leqslant n$.

Consider the above expression $|P_iB|$. Set $i = 0$. For $|P_0B| = |MB|$ we shall use the notation d. It is seen, by Figure 2, that

$$(2) \qquad d = \sqrt{(s\sqrt{2} - r)^2 + h^2}.$$

Our procedure leads to some computational work and to a flowchart (for a computer program), given in Section 3.

3. Deduction of formulas. A flowchart

We shall introduce some variables that depend on i. We abstain from showing this dependence in the notation. (These variables are θ, L, φ, $\Delta\varphi$, ξ, η).

Set $\theta = i \cdot 45^0/n$ and $L = |P_iB|$. Then, by Figures 4 and 2,

$$L = \sqrt{(r \sin \theta)^2 + (s\sqrt{2} - r \cos \theta)^2 + h^2}.$$

Set

(3) $$c_1 = r^2 + 2s^2 + h^2 \quad \text{and} \quad c_2 = 2sr\sqrt{2}.$$

Then

(4) $$L = \sqrt{c_1 - c_2 \cos \theta}.$$

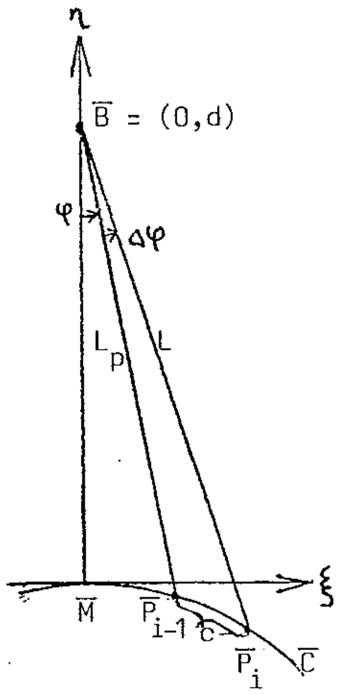
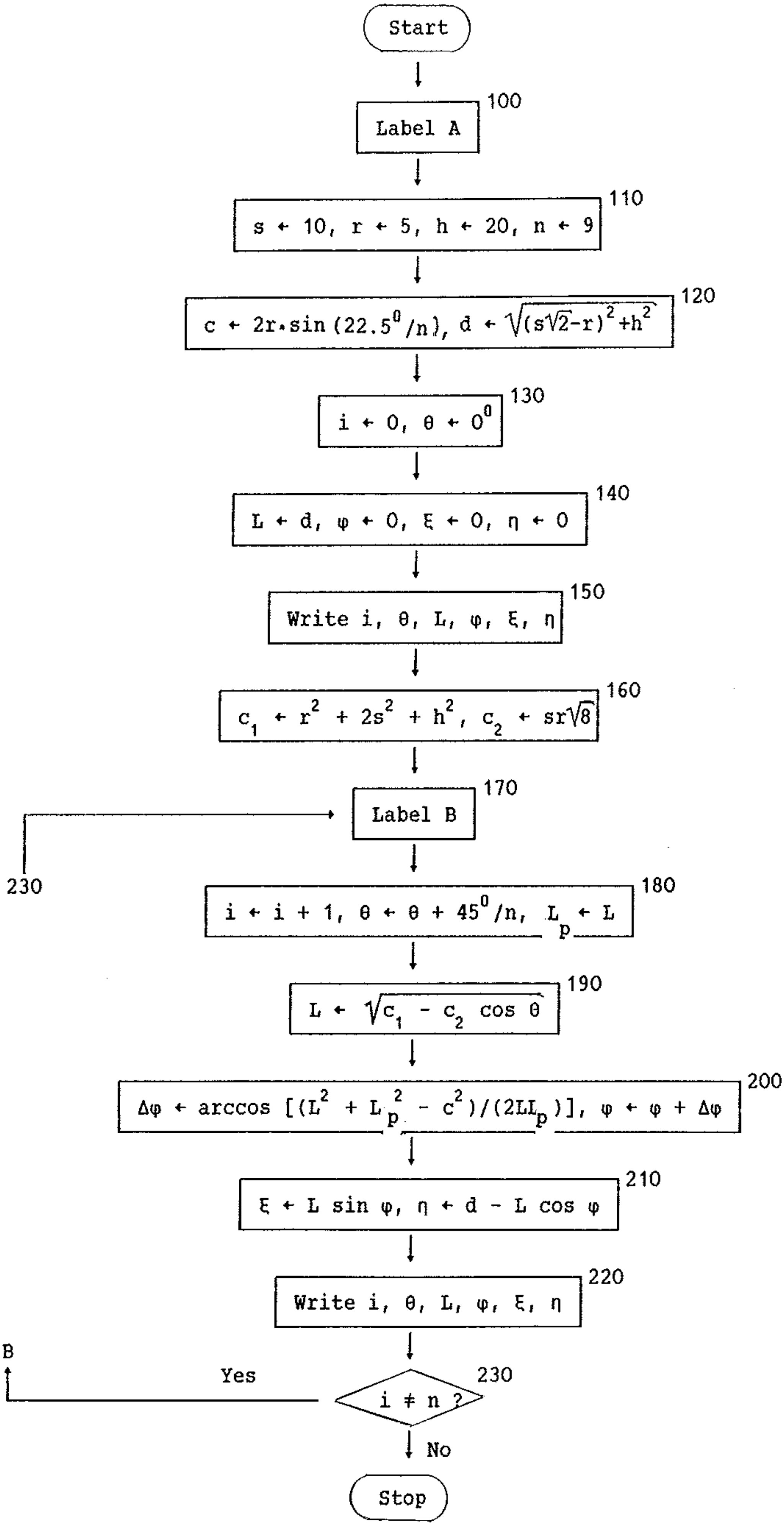


Figure 5. The angles φ and $\Delta\varphi$.

Consider Figure 5 (and compare with Figure 3). We let φ and $\Delta\varphi$ denote the measures of the angles $\overline{M}\overline{B}\overline{P}_i$ and $\overline{P}_{i-1}\overline{B}\,\overline{P}_i$ respectively, and L_p, "L- previous", the length of the line segment $\overline{B}\overline{P}_{i-1}$.

The introduced notation and the equations (1), (2), (3), (4) give the following flowchart. The values assigned to s, r, h, n, in Block 110 are given (as an example) earlier in the text. In Block 200 we apply the law of cosines; the plus sign in the second assignment is given by the inequality in (c). Blocks 170 - 230 form a loop with the counter i.


Start
100
Label A
110
s ← 10, r ← 5, h ← 20, n ← 9
120
c ← 2r·sin (22.5⁰/n), d ← √((s√2-r)² +h²)
130
i ← 0, θ ← 0⁰
140
L ← d, φ ← 0, ξ ← 0, η ← 0
150
Write i, θ, L, φ, ξ, η
160
c1 ← r² + 2s² + h², c2 ← sr√8
170
Label B
230
180
i ← i + 1, θ ← θ + 45⁰/n, Lp ← L
190
L ← √(c1 - c2 cos θ)
200
Δφ ← arccos [(L² + Lp² - c²)/(2LLp)], φ ← φ + Δφ
210
ξ ← L sin φ, η ← d - L cos φ
220
Write i, θ, L, φ, ξ, η
B
Yes
230
i ≠ n ?
No
Stop

4. Output of a computer program

A computer program based on the above flowchart gives the output shown in Table 1.

Table 1. Output of the program (n = 9).

i	θ	L	φ	ξ	η
0	0	21.99	0	0	
1	5	22.00	1.14	0.4361	-0.0079
2	10	22.04	2.27	0.8717	-0.0316
3	15	22.10	3.39	1.3061	-0.0707
4	20	22.18	4.50	1.7389	-0.1248
5	25	22.29	5.59	2.1697	-0.1934
6	30	22.42	6.66	2.5981	-0.2756
7	35	22.56	7.70	3.0238	-0.3705
8	40	22.73	8.72	3.4468	-0.4770
9	45	22.91	9.72	3.8671	-0.5938

How close is the last point, (ξ_n, η_n) (or (ξ_9, η_9)) to the true point (the point $\bar{C}$ in Figure 3)?

To get an answer (at least a partial answer) to this question we run the program with the assignment n ← 9 in Block 110 replaced by n ← 20 and by n ← 45. Then Table 2 is obtained.

Table 2. The last point for some n-values.

n	θ	L	φ	ξ_n	η_n
9	45	22.913	9.7165	3.8671	-0.59377
20	45	22.913	9.7188	3.8680	-0.59361
45	45	22.913	9.7192	3.8681	-0.59359

It is seen that the ξ_n-values for n = 9 and n = 45 differ by around 0.001 cm. Less for the η_n-values. This indicates that the last point converges rapidly when n increases, and that the program, already with the (small) value n = 9, gives an information that is sufficient for practical purposes.

5. A curve close to a circle arc

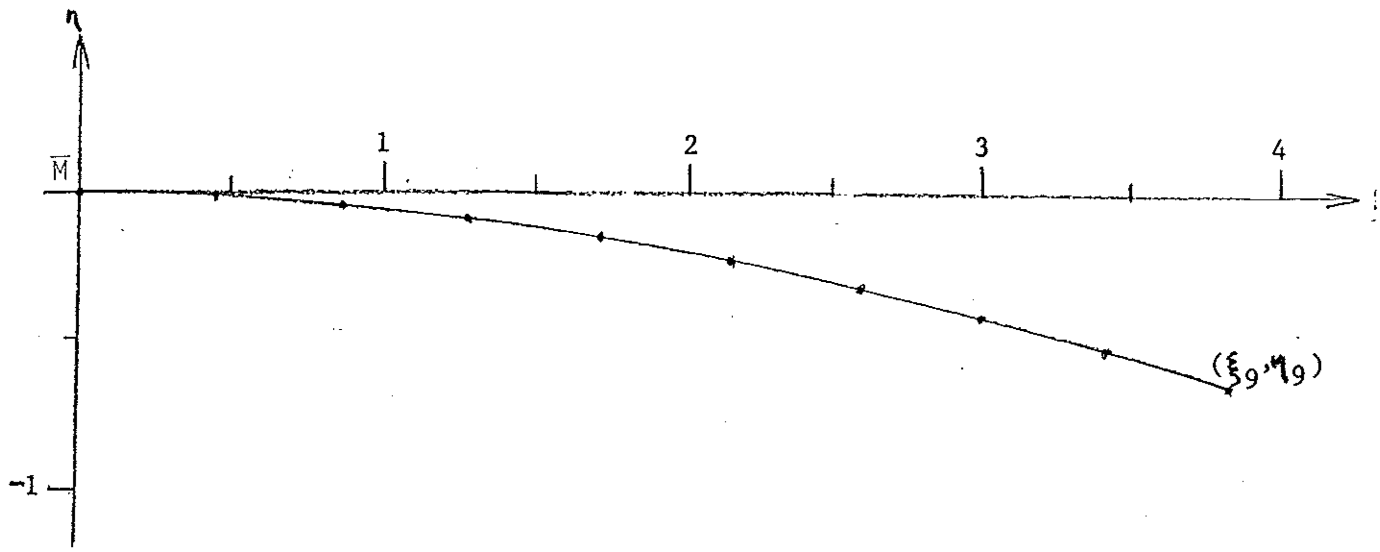


Figure 6. The points (ξ_i, η_i) in Table 1.

Consider Figure 6. It shows points (ξ_i, η_i) given by data in Table 1, and a smooth curve (drawn with help of a French rule) through the points.

The curve seems to be close to some circle arc. How close?

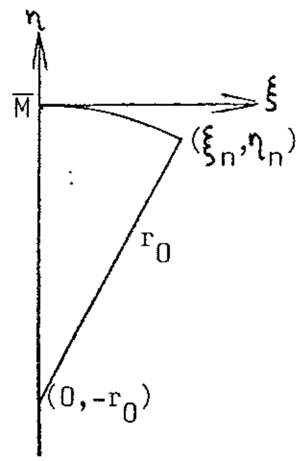


Figure 7. The radius r_0.

To get an answer to this question, we determine a positive number r_0 (see Figure 7) such that

$$\xi_n^{\,2} + (\eta_n + r_0)^2 = r_0^{\,2};$$

here ξ_n, η_n (or ξ_{45}, η_{45}) denote values in the last line of Table 2. We get $r_0 \approx 12.8897$. We then introduce, in the flowchart in Section 3, a Block 215: $r_i \leftarrow \sqrt{\xi^2 + (\eta + r_0)^2}$, and we let Block 220 (also) write $r_i - r_0$. We get Table 3.

Table 3. The deviation $r_i - r_0$ from the circle.

i	$r_i - r_0$
0	0
1	-0.0005
2	-0.0020
3	-0.0043
4	-0.0069

i	$r_i - r_0$
5	-0.0093
6	-0.0108
7	-0.0105
8	-0.0073
9	0

Table 3 shows that the largest value of $|r_i - r_0|$ is about 0.01 cm. This small length can be disregarded when we cut the unfolded cornet out of a metal sheet (so we substitute a circle arc for the curve $\overline{D}\overline{M}\overline{C}$ in Figure 3).

6. Unfolding the sun cell cornet

Consider Figure 8. It shows the triangle ABC of Figure 2, with the coordinates of B and C and the sizes of the side BC and the angle C computed.

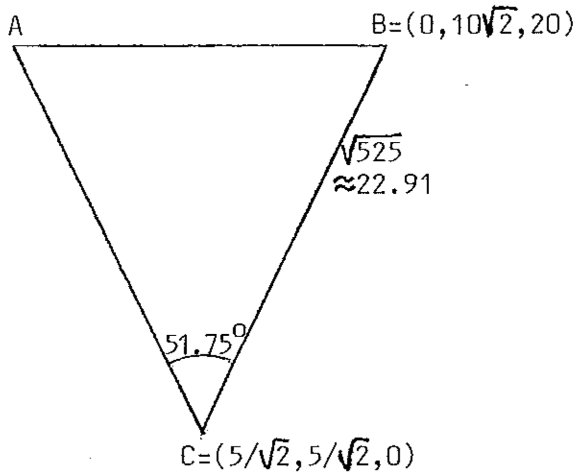


Figure 8. The triangle ABC.

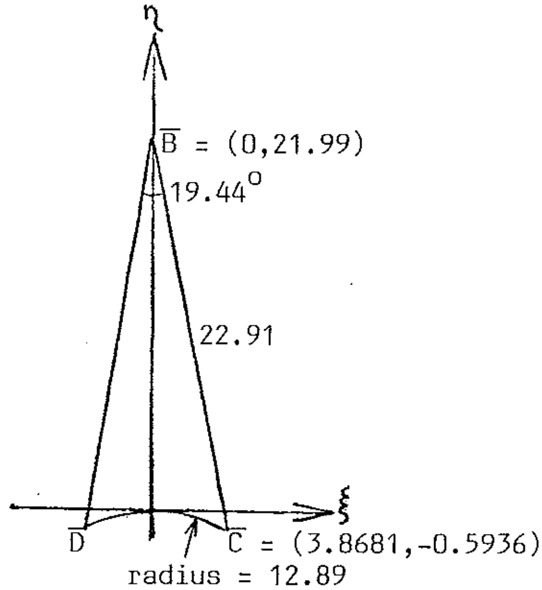


Figure 9. The region $\bar{B}\bar{C}\bar{D}$.

Consider Figure 9. It shows a variant of Figure 3, now letting $\bar{C}\bar{D}$ be a circle arc (as suggested at the end of Section 5). It also shows the length of $\bar{B}\bar{C}$ (equal to the length of BC in Figure 8), the coordinates of $\bar{B} = (0, \sqrt{(10\sqrt{2}-5)^2 + 20^2})$ and $\bar{C}$ (from the last line in Table 2), and the sizes of the angle $\bar{B}$ (from φ in Table 2) and the radius of $\bar{C}\bar{D}$ (from the end of Section 5).

Finally, consider Figure 10. It is composed of four copies of each of the regions in Figures 8 and 9.

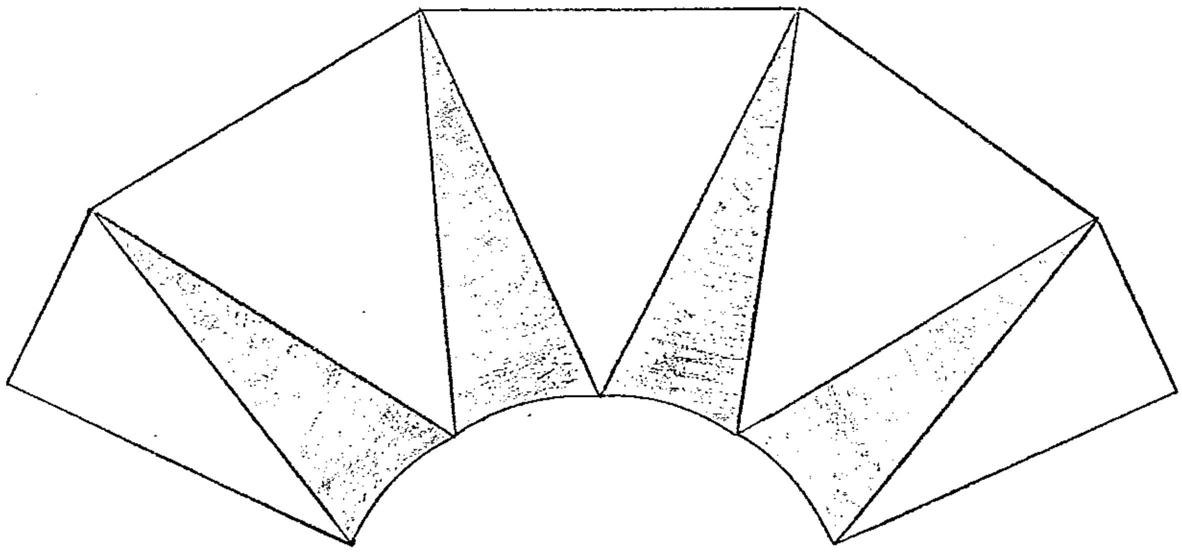


Figure 10. The unfolded cornet.

Figure 10 shows the sun cell cornet of Figures 1 and 2, unfolded on a plane.

7. Reflection in the sun cell cornet

We begin by studying an introductory problem.

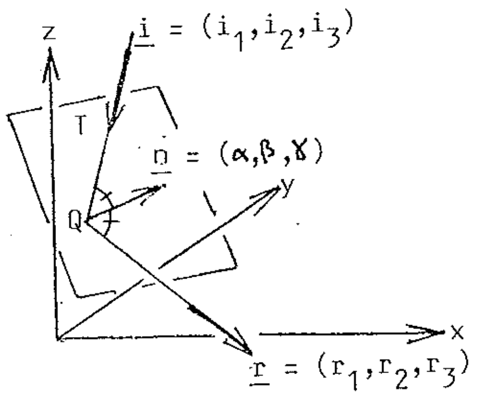


Figure 11. Reflection in a surface.

Assume that a ray, with the direction of the vector $\underline{i}$, is incident at a point Q of a surface that has, at Q, a tangent plane T (see Figure 11) with the equation $\alpha x + \beta y + \gamma z + \delta = 0$. Find a vector $\underline{r}$ with the direction of the reflected ray, in terms of $\underline{i}$, α, β, γ.

We also assume that

(5) $$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

and that $\underline{r}$ and $\underline{i}$ have the same length. (These two assumtions simplify our solution.) We observe that $\underline{n} = \overrightarrow{(\alpha, \beta, \gamma)}$ is a unit normal vector of T. We denote the components of $\underline{i}$ and $\underline{r}$ as in Figure 11. The vector $\underline{r} - \underline{i}$ is parallel to the vector $\underline{n}$, and we have for two dot products: $\underline{n} \cdot \underline{r} = -\underline{n} \cdot \underline{i}$. Hence,

(d) $$\frac{r_1 - i_1}{\alpha} = \frac{r_2 - i_2}{\beta} = \frac{r_3 - i_3}{\gamma},$$

(e) $$\alpha r_1 + \beta r_2 + \gamma r_3 = -(\alpha i_1 + \beta i_2 + \gamma i_3).$$

Denote by -k the common value of the three fractions in (d). Then formulas (d) and (e) show that

$$\alpha(i_1 - \alpha k) + \beta(i_2 - \beta k) + \gamma(i_3 - \gamma k) = -(\alpha i_1 + \beta i_2 + \gamma i_3),$$

(6) $$k = 2(\alpha i_1 + \beta i_2 + \gamma i_3), \quad \text{and}$$

(7) $$\overrightarrow{(r_1, r_2, r_3)} = \overrightarrow{(i_1 - \alpha k, i_2 - \beta k, i_3 - \gamma k)}.$$

Summing up: The equation (7) solves the introductory problem, with k defined in (6). When (7) is used, the equation of the plane T must be normalized so that (5) holds.

We want to study reflection in the sun cell cornet of Figures 1 and 2. It is then, for reasons of symmetry, sufficient to consider two special cases.

First, assume that Q belongs to the planar region within the triangle ABC in Figure 2. We have, in coordinate form,

$$A = (s\sqrt{2},\ 0,\ h),\quad B = (0,\ s\sqrt{2},\ h),\quad C = (r/\sqrt{2},\ r/\sqrt{2},\ 0).$$

Hence T is the plane

$$hx + hy - \sqrt{2}(s-r)z - \sqrt{2}rh = 0.$$

We have for the coefficients of x, y, z:

$$\sqrt{h^2 + h^2 + [\sqrt{2}(s-r)]^2} = \sqrt{2}\sqrt{h^2 + (s-r)^2}.$$

A unit normal vector of the plane then is

(8) $$\overrightarrow{(\alpha,\ \beta,\ \gamma)} = \overrightarrow{(h/\delta,\ h/\delta,\ -\sqrt{2}(s-r)/\delta)}$$

where

(9) $$\delta = \sqrt{2}\sqrt{h^2 + (s-r)^2}.$$

The formulas (9), (8), (6), (7) can be used to handle reflection in the region ABC.

Second, assume that Q belongs to the curved region within the contour BCD in Figure 2. Let P (see Figure 2) be the point where the straight line BQ intersects the xy-plane. Let $\underline{t}$ be a tangent vector of the circle in Figure 2 at the point P, viz. (defining θ as in Figure 4) $\underline{t} = \overrightarrow{(\cos\theta,\ -\sin\theta,\ 0)}$. A normal vector (of the curved region BCD) at Q is perpendicular to each of the vectors $\underline{t}$ and $\overrightarrow{BP}$. Such a normal vector is the cross product

$$\underline{t} \times \overrightarrow{BP} = \overrightarrow{(\cos\theta,\ -\sin\theta,\ 0)} \times \overrightarrow{(r\sin\theta,\ r\cos\theta - s\sqrt{2},\ -h)} =$$
$$\overrightarrow{(h\sin\theta,\ h\cos\theta,\ r - s\sqrt{2}\cos\theta)}.$$

We divide the components of this last vector by the length of the vector (i.e., by $\sqrt{h^2 + (r - s\sqrt{2}\cos\theta)^2}$) to get a unit normal vector, $\overrightarrow{(\alpha,\ \beta,\ \gamma)}$, of the region BCD at the point Q. The formulas (6) and (7) can now be used to handle reflection in the region BCD.

Appendix: A uniqueness property

We introduce some terms that are convenient for our discussion. (The terms are: divide, convex, generator, position.)

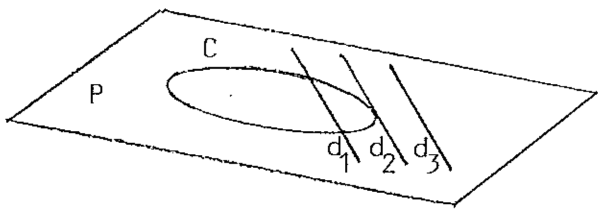


Figure 12. Dividing a planar set.

Assume that C is a point set (a curve, for example) and that d is a straight line, both situated in a plane P. We shall say that d divides C, if C has points at each side of d. An example: If C is a circle, d_1 a straight line that intersects C at two points, d_2 a tangent line, and d_3 a straight line in P outside C (see Figure 12), then d_1 divides C, and d_2 and d_3 do not. We shall also say that a plane that intersects P along a dividing line d divides C. If there is, through each point of C, a straight line in P that does not divide C, we shall call C a convex set.

Let S be a point set (a surface, for example) and Q a plane, both in xyz-space. If S has points at each side of Q, we shall say that Q divides S. If there is, through each point of S, a plane that does not divide S, we shall call S a convex set.

Now let S be a solution (in xyz-space) of the problem in Section 1. (Then S can be unfolded on a plane, the $\xi\eta$-plane.) Let EF be a line segment

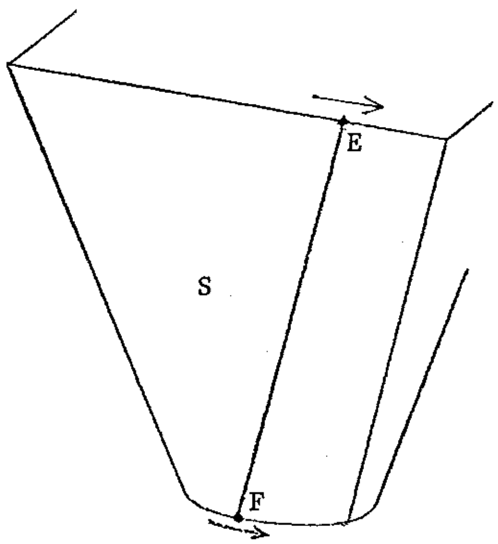


Figure 13. The generator EF.

(of variable length) that moves around S (see Figure 13), so that its end points E and F move around the square and the circle (of Figure 1) respectively, and that each location of EF hits the $\xi\eta$-plane (along a line segment) during the unfolding. We shall call EF the generator of S, and each location of EF a position of EF.

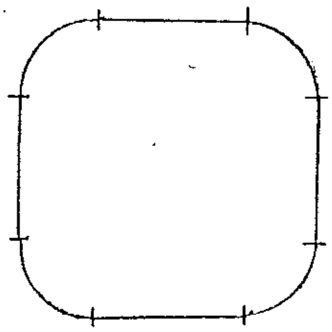


Figure 14. Intersection of S_0 and $z = h_1$.

Let S_0 denote the surface introduced in Section 2 (cf. Figure 2). Assume that $0 < h_1 < h$, and consider the intersection of S_0 and the plane $z = h_1$. Figure 14 shows such an intersection. It consists of four line segments and four circle arcs. Is is seen that the intersection is a convex curve. It is also seen that S_0 is a convex surface.

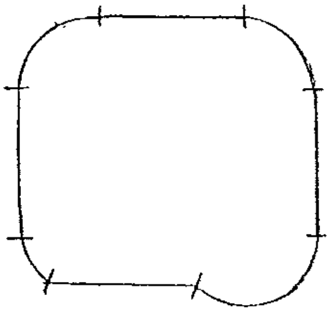


Figure 15. Intersection of S_1 and $z = h_1$.

Now make some changes in Figure 2. Replace the point C by the point $C_1 = (r\sqrt{3}/2,\ r/2)$ and the triangle ABC by the triangle ABC_1. Also replace the curved region BCD by the larger region BC_1D and the curved region having the boundary point $(r,0,0)$ by a smaller region (chosen in an obvious way). Let S_1 denote the surface obtained. It is seen that S_1 is a solution of our problem, and that S_1 intersects the plane $z = h_1$ along a curve as in Figure 15. This curve is not convex. Neither is S_1 a convex surface.

Using the idea for the surface S_1, we can of course construct infinitely many non-convex solutions of the problem.

Does the problem have any other convex solution than S_0?

To answer this question, assume that S is a solution distinct from S_0. The generator of S then has a position, we denote it EF, such that EF is not a position of the generator of S_0. Assume that E is a point of the side AB of the square in Figure 2, the point A excluded and the point B included. (This is an inessential restriction, for reasons of symmetry.)

First, assume that B and E are distinct points. Then the points C (of Figure 2) and F are distinct. It is seen that each plane through EF divides the square or the circle (of Figure 2) or both. It follows that the surface S is non-convex.

Second, assume that the point E coincides with B. Then F is situated outside the 90^0 arc CD of the circle (in Figure 2). It is seen that each plane through EF divides the square or the circle or both, and that, hence, S is non-convex also in this case.

There follows a uniqueness property:

The problem in Section 1 has precisely one convex solution: The surface S_0.

Remark 1. Let a straight line glide along the square and the circle of Figure 2, so that it all the time intersects the negative z-axis. Do we get one more solution of the problem in Section 1?

This question has the answer no, for it is known from differential geometry (see [1], pp. 30-32) that if a surface can be unfolded on a plane and if it has tangent planes at points of a position of the generator, then these tangent planes are identical.

Remark 2. Let us say that a surface is smooth, if it has a tangent plane at each interior point. It is seen that the surfaces S_0 and S_1 (see the context of Figures 14 and 15) are smooth and non-smooth respectively.

Does the problem of Section 1 have any other smooth solution than S_0?

This question has the answer yes.

We abstain from motivating the answer here, for the smooth solutions (except S_0) that we have found are quite complicated (and of no interest in practice).

Reference

[1] DETLEF LAUGWITZ, Differentialgeometrie, B.G.Teubner Verlagsgesellschaft, Stuttgart 1960.