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Refraction in spacetime

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Refraction, interference, and diffraction are distinguishing features of wave-like phenomena. Although they are usually associated only with a purely spatial wave-propagation pattern, analogs to interference and diffraction involving the spatio-temporal dynamics of waves in one dimension have been discussed. We complete the triplet of analogies by discussing how spatio-temporal analogs to refraction are exhibited by a quantum particle in one dimension that is scattering off a step barrier. Similarly, birefringence in spacetime occurs for a spin-1/2 particle in a magnetic field.

I. INTRODUCTION

Quantum mechanics is essentially a wave theory, and analogs with classical optics can be fruitful for providing a deeper understanding of quantum phenomena. The analogy between the Helmholtz equation and the time-independent Schrödinger equation leads to the existence of identical phenomena in classical optics and quantum mechanics for purely static situations.[1] The three archetypical wave-like phenomena encountered in undergraduate textbooks are diffraction around sharp corners, interference of waves propagating along different paths, and refraction at boundaries between regions with different wave speeds.

There has been a growing interest in quantum dynamics, and time-dependent treatments of quantum mechanics have found their way into textbooks.[2–4] Computerized studies of wave-packet scattering were pioneered in 1967,[5] and have revealed nonintuitive features.[6,7]

For quantum dynamics in one spatial dimension, an exact analogy with stationary diffraction in optics was found by Moshinsky in 1952 in his work on diffraction in time;[8] see also Ref. 1. Since then, the field has expanded considerably.[9] Experiments with ultracold atoms have observed interference in time,[10] and more recently, interference in time without spatial dynamics has been shown to give rise to interference effects in the time-energy domain.[11]

In this paper we complete the set of spatio-temporal analogues of wave phenomena by discussing the occurrence of refraction in spacetime for wave-packet propagation through a one-dimensional inhomogeneous medium. As a further application, the case of birefringence[12] is considered whose spatio-temporal analog appears in the propagation of a wave packet that is an equal superposition of spin-up and spin-down states through a region with a spatially inhomogeneous magnetic field.

The remainder of this article is organized as follows. In Sec. II we provide background information and establish our notation by discussing stationary refraction in space for the solutions of electromagnetic and Schrödinger wave equations. The emergence and properties of refraction in spacetime are explained based on a ray picture in Sec. III. Simulations of wave-packet dynamics are presented in Sec. IV to show the validity of the basic ray model for refraction in spacetime, and to illustrate regimes where it needs to be modified and breaks down. Section V is devoted to the phenomenon of birefringence exhibited by spin-1/2 particles entering a region of finite and spatially uniform magnetic fields from a field-free region. Further discussion and conclusions are presented in Sec. VI.

II. ORDINARY REFRACTION OF CLASSICAL AND QUANTUM-PROBABILITY WAVES

Each Cartesian component of the electromagnetic field in a dielectric medium satisfies a wave equation whose solutions can be expressed as a superposition of single-frequency wave amplitudes $\Psi_{\rm e}(\mathbf{x},t) = \psi_{\rm e}(\mathbf{x})\,{\rm e}^{-i\omega t}$. The stationary wave amplitude is found from the Helmholtz equation

$$\boldsymbol{\nabla}^2\psi_{\rm e}(\mathbf{x}) + k^2\psi_{\rm e}(\mathbf{x}) = 0, \tag{1}$$

where k denotes the wave number. In vacuum, $k = k_0 \equiv \omega/c_0$, where $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum. The effect of a medium is described by the refractive index $n_{\rm e} = k/k_0$ which, in general, is a frequency-dependent materials parameter.

For stationary situations the optical case is analogous to quantum mechanics. Consider the time-dependent Schrödinger equation for a massive particle subject to an external potential $V(\mathbf{x})$, which is given by

$$i\hbar\frac{\partial\Psi_{\rm q}(\mathbf{x},t)}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\Psi_{\rm q}(\mathbf{x},t) + V(\mathbf{x})\,\Psi_{\rm q}(\mathbf{x},t). \tag{2}$$

We can make the separation *ansatz* $\Psi_{\rm q}(\mathbf{x},t) = \psi_{\rm q}(\mathbf{x})\,{\rm e}^{-iEt/\hbar}$ and find the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi_{\rm q}(\mathbf{x}) + V(\mathbf{x})\,\psi_{\rm q}(\mathbf{x}) = E\,\psi_{\rm q}(\mathbf{x}), \tag{3}$$

or

$$\boldsymbol{\nabla}^2\psi_{\rm q}(\mathbf{x}) + \frac{2m}{\hbar^2}\left[E - V(\mathbf{x})\right]\psi_{\rm q}(\mathbf{x}) = 0, \tag{4}$$

which has the same form as the classical Helmholtz equation (1). The formal analogy[13,14] with the optical case is complete for a constant potential $V(\mathbf{x}) \equiv V_0 \leq E$. This analogy can be made explicit by defining $k = \sqrt{2m(E-V_0)/\hbar^2}$ and $\omega = E/\hbar$. (We consider here only propagating solutions corresponding to real k.) The potential V_0 determines the wave number, and the quantity

$$n_{\mathrm{q}} = \sqrt{1 - \frac{V_0}{\hbar\omega}} \tag{5}$$

is the quantum-mechanical equivalent of the refractive index in optics.[15] With this definition of n_{q}, the solutions for classical optics are identical to the ones for quantum optics whenever n_{e} is identical to n_{q}.

The Helmholtz equation has plane-wave solutions of the form $\psi = \psi^{(0)}\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}$. Here $\mathbf{k}$ denotes the wave vector with magnitude $|\mathbf{k}| = k$ and direction $\hat{\mathbf{k}}$ coinciding with the propagation direction of the wave, and $\psi^{(0)}$ is a constant. When such a plane wave is incident at a nonzero angle on the interface between two media with different refractive indices, a change of direction is observed for the part of the wave passing from one medium into the other. This phenomenon of refraction is illustrated in Fig. 1(a). Because both the wave frequency and wave-vector component parallel to the interface are not changed in this process, the relation

$$k^{(1)} \sin\theta^{(1)} = k^{(2)} \sin\theta^{(2)} \tag{6}$$

holds. In terms of the refractive indices in the two adjoining spatial regions, Eq. (6) can be written as

$$n_{\mathrm{e,q}}^{(1)} \sin\theta^{(1)} = n_{\mathrm{e,q}}^{(2)} \sin\theta^{(2)}. \tag{7}$$

Because we consider electromagnetic waves as the example for the classical case, the refractive index is a property of the dielectric media occupying the spatial regions $x > 0$ and $x < 0$, respectively. For the quantum case the refractive indices are defined in accordance with Eq. (5) for a piecewise constant external potential

$$V(x) = \begin{cases} V_0^{(1)} & (x < 0) \\ V_0^{(2)} & (x > 0). \end{cases} \tag{8}$$

Equation (7) is the familiar law of refraction. The case with $\theta^{(1)} < \theta^{(2)}$ shown in Fig. 1(a) corresponds to $n^{(1)} > n^{(2)}$ as illustrated in Fig. 1(d). In quantum mechanics this case would be realized by having $V_0^{(1)} < V_0^{(2)}$. In general, only part of the wave is transmitted through the interface and undergo refraction, while the other part is reflected back into the medium from where the wave is incident. To properly describe both reflection and refraction at an interface, the Fresnel equations have to be used.[16]

III. REFRACTION IN $1+1$ DIMENSIONS

We now consider the space-time description of rays in an inhomogeneous medium in one spatial dimension and elucidate the phenomenon of refraction in spacetime. To define angles and trigonometric functions in the plane spanned by the time and spatial coordinates, these two quantities need to have identical dimensions. The simplest way to do so is to multiply time by a constant reference speed, turning it into an equivalent distance, $t \to v_0 t$. For optical waves a natural choice is $v_{0,\mathrm{e}} = c_0$, thus giving the time axis the meaning of propagated distance in empty space. In the same spirit, we use $v_{0,\mathrm{q}} = \sqrt{2E/m}$ for quantum-probability waves.

In a homogeneous one-dimensional medium, the speed of a wave packet is its local group velocity $v_{\mathrm{g}} = \partial\omega/\partial k$, and the slope of its worldline in the $v_0 t$-x diagram is given by $\Delta(v_0 t)/\Delta x = \tan\vartheta$. We use ϑ to denote the angle in the spacetime diagram, rather than θ, which we use for spatial refraction. The constant flow of time in two adjoining regions 1 and 2 where the group velocity is different implies the occurrence of refraction in the $v_0 t$-x space described by the relation between incident and transmitted angles [see Fig. 1(b)]

$$v_{\mathrm{g}}^{(1)} \tan\vartheta^{(1)} = v_{\mathrm{g}}^{(2)} \tan\vartheta^{(2)}. \tag{9}$$

If we substitute the definition of the refractive index into the expression for the group velocity, we obtain the general relation

$$v_{\mathrm{g}} = \frac{v_0}{n}\left(1 + \alpha_v \frac{\omega}{n}\frac{\partial n}{\partial\omega}\right)^{-1}, \tag{10}$$

where $\alpha_v = v_0 k_0/\omega$ is the ratio of the group and phase velocities in the medium with $n = 1$ (that is, vacuum). We combine Eqs. (9) and (10) and find

$$n^{(2)}\left(1 + \alpha_v \frac{\omega}{n^{(2)}}\frac{\partial n^{(2)}}{\partial\omega}\right)\tan\vartheta^{(1)} = n^{(1)}\left(1 + \alpha_v \frac{\omega}{n^{(1)}}\frac{\partial n^{(1)}}{\partial\omega}\right)\tan \tag{11}$$

as the relation describing refraction in spacetime. For an electromagnetic wave crossing between nondispersive media (that is, constant $n_{\mathrm{e}}^{(1,2)}$), Eq. (11) reduces to

$$n_{\mathrm{e}}^{(2)} \tan\vartheta_1 = n_{\mathrm{e}}^{(1)} \tan\vartheta_2. \tag{12a}$$

This case is depicted in Fig. 1(c). A different result is obtained in the quantum case where the refractive index is frequency-dependent [see Eq. (5)], yielding

$$n_{\mathrm{q}}^{(1)} \tan\vartheta_1 = n_{\mathrm{q}}^{(2)} \tan\vartheta_2. \tag{12b}$$

The ray picture for this case is illustrated in Fig. 1(b).

We have found a refraction-like phenomenon in the spatio-temporal evolution of a wave propagating in one spatial dimension. The kinematic differences associated with refraction in spacetime compared to ordinary refraction (in space) are captured in Eqs. (9) and (6). Equation (7) holds for refraction in space for any type of wave because refractive indices are defined in terms of the wave number (or phase velocity) and not the group velocity. The latter determines refraction in spacetime. Hence the

1 + 1-dimensional equivalent of Eq. (11) is not simply determined by the values of the refractive indices, but is also influenced by their frequency dependence. As a result, spacetime-refracted beams can be bent in opposite directions for waves that are subject to the same spatial profile of the refractive index, as is illustrated in Figs. 1(b) and 1(c).

So far, we have stressed the wave properties and refraction angles in spacetime, without referring to the associated particle properties exhibited in scattering events. Within the particle picture, the slope of the worldline in a spacetime diagram corresponds to the particle's velocity. Thus in the case depicted in Fig. 1(b), which applies to refraction with $\vartheta_2 > \vartheta_1$, the particle speeds up after the discontinuity.[17]

IV. REFRACTION IN SPACETIME EXHIBITED IN QUANTUM WAVE-PACKET DYNAMICS

In Sec. III we considered plane-wave solutions to the time-independent wave equations. To make our results more visualizable, we now turn to the dynamics of wave packets in one spatial dimension. We can superimpose a continuum of scattering solutions to construct the wave function corresponding to any sufficiently smooth initial state. As a model system, we choose the example of scattering off a finite potential step given by Eq. (8) with $V_0^{(1)} = 0$ and $V_0^{(2)} = V_0$. For a incident wave from region 1, the stationary wave functions are

$$\psi_k(x) = \begin{cases} \mathrm{e}^{ikx} + r(k)\,\mathrm{e}^{-ikx} & (x < 0) \\ t(k)\,\mathrm{e}^{ik'x} & (x > 0) \end{cases} \tag{13}$$

Here $k = \sqrt{2mE/\hbar^2}$, $k' = \sqrt{2m(E - V_0)/\hbar^2}$, and the reflection and transmission amplitudes are (see, for example, Ref. 2)

$$r(k) = \frac{k - k'}{k + k'} \qquad t(k) = \frac{2\sqrt{kk'}}{k + k'}. \tag{14}$$

A free Gaussian wave packet centered at $x = x_0$ and evolving in time can be expressed as a linear combination of plane waves,

$$\psi_{\mathrm{G}}(x,t) = \int_{-\infty}^{\infty} dk\, \mathrm{e}^{-ik(x-x_0)}\, \mathrm{e}^{-\frac{(k-k_0)^2}{2\Delta k^2}}\, \mathrm{e}^{-i\frac{\hbar k^2}{2m}t}, \tag{15}$$

Similarly, its dynamics when scattering off a potential step is given by the analogous superposition of solutions ψ_k:

$$\psi(x) = \int_{-\infty}^{\infty} dk\, \psi_k(x)\, \mathrm{e}^{ikx_0} \mathrm{e}^{-\frac{(k-k_0)^2}{2\Delta k^2}}\, \mathrm{e}^{-i\frac{\hbar k^2}{2m}t}. \tag{16}$$

In Fig. 2 we show the dynamics of a Gaussian wave packet scattering off a potential step, with worldlines for the incoming, reflected, and refracted trajectories superimposed. The parameters used were chosen such that $|r(k)|^2 \approx 0.5$, which ensures sufficient visibility of the fringes arising from interference between incoming and reflected parts of the wave packet. The transmitted wave is represented by a ray with larger slope (i.e., smaller group velocity) than the rays associated with the incident and reflected partial waves because the scattering occurs against the positive potential step. In terms of our discussion of refraction in spacetime, Fig. 2 corresponds to the scenario illustrated in Fig. 1(b), with the worldline of the transmitted beam being refracted away from the normal as scattering for the quantum wave occurs against a medium with a lower index of refraction.

For the ray picture to properly describe the propagation of a wave packet, the latter should not spread appreciably during the scattering process to avoid effects due to broadening. Free wave packets broaden due to dispersive effects from the initial distribution of momenta. This broadening will become noticeable over a time scale

$$t_{\mathrm{broad}} \approx \frac{m\Delta x^2}{\hbar}. \tag{17}$$

The time scale for broadening needs to be compared with the time needed for wave propagation over a characteristic distance L, for example, the system size,

$$t_{\mathrm{prop}} = \frac{mL}{\hbar k}. \tag{18}$$

Effects due to dispersion will be important when $t_{\mathrm{broad}} \lesssim t_{\mathrm{prop}}$. This condition can be expressed as

$$1 \gtrsim \frac{\Delta x^2}{\lambda L} \equiv F, \tag{19}$$

where the parameter F is the same form as the Fresnel parameter, which distinguishes near-field and far-field diffraction in optics.[16] A situation with $F < 1$ is shown in Fig. 3, where refraction in spacetime for a Gaussian wave packet scattering off a step potential is shown. In contrast to Fig. 2, the wave packet is seen to broaden significantly. In addition, the interference fringes are curved and exhibit some diffraction in time.[18] Figure 3 thus exhibits all three wave phenomena: refraction at the boundary, interference between the incident and reflected components, and diffraction as the fringes curve. The last of these effects is due to the presence of a broad range of momentum components in the initial wave packet, causing considerable broadening during the scattering event, which is displayed in spacetime as bending of the interference fringes.

V. BIREFRINGENCE IN SPACETIME

Birefringence is a property of materials that exhibit two distinct optical densities, for example, calcite crystals, and is routinely used in undergraduate physics laboratories.[19,20] Depending on its polarization, light

moves through the crystal in two different paths, resulting in a double image of the original light beam. The material has two indices of refraction, $n_\parallel$ and $n_\perp$.

Here we consider a spin-1/2 particle in a magnetic field as an analogy. The quantization axis for spin is naturally chosen to be parallel to the field direction. Due to the Zeeman effect, the particle experiences a spin-dependent potential $s\mu B$, where $s = -(+)1$ for spin-$\uparrow$ ($\downarrow$) states that are parallel (antiparallel) to the field direction. The shape of the spin-dependent potential is the same regardless of the direction of the external field relative to the propagation axis for neutral particles.[21]

The wave function of a particle is given by the spinor

$$\Psi(x,t) = \begin{pmatrix} \Psi_\uparrow(x,t) \\ \Psi_\downarrow(x,t) \end{pmatrix}. \tag{20}$$

The time-dependent Schrödinger equation takes the form

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + H_z \Psi(x,t), \tag{21}$$

where the effect of the magnetic field is included in the Zeeman matrix

$$H_z = \begin{pmatrix} -\mu B & 0 \\ 0 & \mu B \end{pmatrix}. \tag{22}$$

Because the Zeeman term (22) is diagonal, Eq. (21) separates into two distinct equations for the individual spin components,

$$i\hbar \frac{\partial \Psi_{\uparrow/\downarrow}(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi_{\uparrow/\downarrow}(x,t)}{\partial x^2} \mp \mu B \Psi_{\uparrow/\downarrow}(x,t), \tag{23}$$

allowing the problem to be treated as the superposition of two independent propagations. In Fig. 4 the probability density $|\Psi_\uparrow(x,t)|^2 + |\Psi_\downarrow(x,t)|^2$ is shown as a function of space and time for a Gaussian wave packet incident from a region with $B = 0$ onto a region with $B > 0$. The initial spin populations were set equal. As the wave packet propagates past the discontinuity at $x = 0$, it splits up into two parts with different velocities as the spinor components refract from opposite potential steps due to the Zeeman splitting. In spacetime, the worldlines of the two spin-components separate as a result of the birefringent refraction. Mechanically, this separation arises from one component being slowed down and the other speeding up after the discontinuity.

It appears from Fig. 4 that the wave packets corresponding to the two opposite-spin components have different widths in region 2 where B is finite. To discuss this phenomenon, we introduce the scattering time Δt as the time it takes for the wave packet to pass from the field-free region 1 to region 2. This transition can be defined by the central part of the wave packet moving over the distance Δx at the discontinuity. We have

$$\frac{\hbar}{m} k \Delta t_1 = \Delta x_1 \tag{24}$$

for the incoming wave packet. Similarly, in region 2, we have

$$\frac{\hbar}{m}\sqrt{k^2 + \frac{2m\mu B}{\hbar^2}}\Delta t_{2,\uparrow} = \Delta x_{2,\uparrow} \tag{25}$$

and

$$\frac{\hbar}{m}\sqrt{k^2 - \frac{2m\mu B}{\hbar^2}}\Delta t_{2,\downarrow} = \Delta x_{2,\downarrow}. \tag{26}$$

Because the scattering time is the same for all three partial waves, a relation must hold between the incident and transmitted wave-packet widths. This relation is given by

$$\frac{\Delta x_{2,\uparrow/\downarrow}}{\Delta x_1} = \sqrt{1 \pm \frac{\mu B}{\hbar^2 k^2}}. \tag{27}$$

We see that not only is the central velocity affected by the potential step, but also the width of the wave packet changes during scattering at the interface.

VI. DISCUSSION AND CONCLUSIONS

The spatio-temporal evolution of a Gaussian wave packet as it scatters over a potential step represents an analogy to stationary wave refraction. Inspired by similar phenomena, we have coined the term *refraction in spacetime* for this effect. The spatio-temporal equivalent of the law of refraction is given in Eq. (11).

Analogies[14] are commonly used in physics as a tool to aid discovery and in the classroom to compare one situation to another efficiently.[22,23] Successful application of an analogy succeeds in conveying information about an unfamiliar topic by relating it to one already known by the learner. To avoid misconceptions, the limitations of any analogy must be made explicit. Additional understanding can sometimes be gained by elucidating the limits of an analogy.

For classical optics and quantum mechanics to be formally analogous in the spatial domain, the refractive index in the quantum case must be defined by Eq. (5). This choice does not imply that the two theories also behave the same in spacetime. As it turns out, refraction in spacetime for a quantum wave behaves like its purely spatial counterpart in the sense that the angle of refraction increases when the index of refraction decreases. The opposite behavior holds for optical waves in nondispersive media in the spatio-temporal domain. This different behavior arises as a consequence of the differing dispersion properties associated with the two cases because the equivalent of law for refraction in spacetime [Eq. (11)] involves not only the refractive indices but also their derivatives with respect to frequency. Conversely, wave phenomena characterized by refractive indices that have the appropriate frequency dependence are formally analogous in both the purely spatial and spatio-temporal domains. In particular, the refractive behavior in $1+1$ dimensions exhibited by quantum waves can be faithfully reproduced by electromagnetic waves propagating

through a junction between two waveguides.[24] We suspect that associated diffraction-in-time effects (as seen in Fig. 3) will be similarly exhibited in both systems. Moreover, light trapped in a suitably designed microcavity behaves identically to matter (quantum) waves[25] and can therefore be expected to exhibit analogous refraction and diffraction phenomena.

Our study of refraction in spacetime illustrates the general importance of dispersion for shaping the nature of spatio-temporal analogies for wave phenomena. As is well known, diffraction in spacetime is absent for electromagnetic waves in vacuum but becomes possible in a dispersive medium.[1] Alternatively, a spatially varying refractive index for optical waves results in a Schrödinger-type equation in the beam-propagation direction instead of physical time,[26] thus allowing us to explore analogous wave phenomena on yet another level.

Acknowledgments

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[12] In traditional optics, birefringence occurs when light propagates through a material that has an index of refraction $n_{\parallel}$ for the components of light with a given polarization, and a different index of refraction for the perpendicular component. As a consequence, the wave is split into two components.

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[17] As an aside, we point out that the phase and group velocities associated with quantum waves show different trends when crossing between regions of space with different potentials. We define the phase velocity $v_{\rm p,q} = \omega/k$ in the usual fashion and find $v_{\rm p,q} = v_{0,\rm q}/2n_{\rm q}$, whereas the group velocity is $v_{\rm g,q} = v_{0,\rm q}n_{\rm q}$. Hence, in a case where the particle (group) velocity increases upon transmission, the phase velocity will decrease (and *vice versa*). This behavior is a consequence of dispersion. More details about quantum phase velocity in situations with a finite external potential are discussed, for example, in L. Bergmann and C. Schaefer, *Optics of Waves and Particles* (de Gruyter, Berlin, 1999), p. 965, and K. U. Ingard, *Fundamentals of Waves and Oscillations* (Cambridge University Press, Cambridge, UK, 1988), Sec. 16.5.

[18] The interference fringes seen in Fig. 2 are parallel to the time axis, and their only time dependence is a weak modulation due to the propagating envelopes of the incoming and reflected wavepackets. Hence, they can be understood as a purely spatial phenomenon. In contrast, the fringes in Fig. 3 exhibit an oscillation in time for fixed x. This effect, which can be attributed to diffraction in spacetime, changes the character of the interference fringes from being spatial in nature to having an additional temporal dimension.

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[24] The refractive index for classical electromagnetic waves propagating in a waveguide is given by $n_{\rm wg} = \sqrt{1-(\Omega^2/\omega^2)}$, where Ω is the cut-off frequency for a particular mode. See, for example, J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley & Sons, Hoboken, NJ, 1999), Sec. 8.3. Specializing the law for refraction in spacetime [Eq. (11)] to this case yields the same expression in terms of refractive indices as is found for quantum waves [Eq. (12b)].

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Figure captions

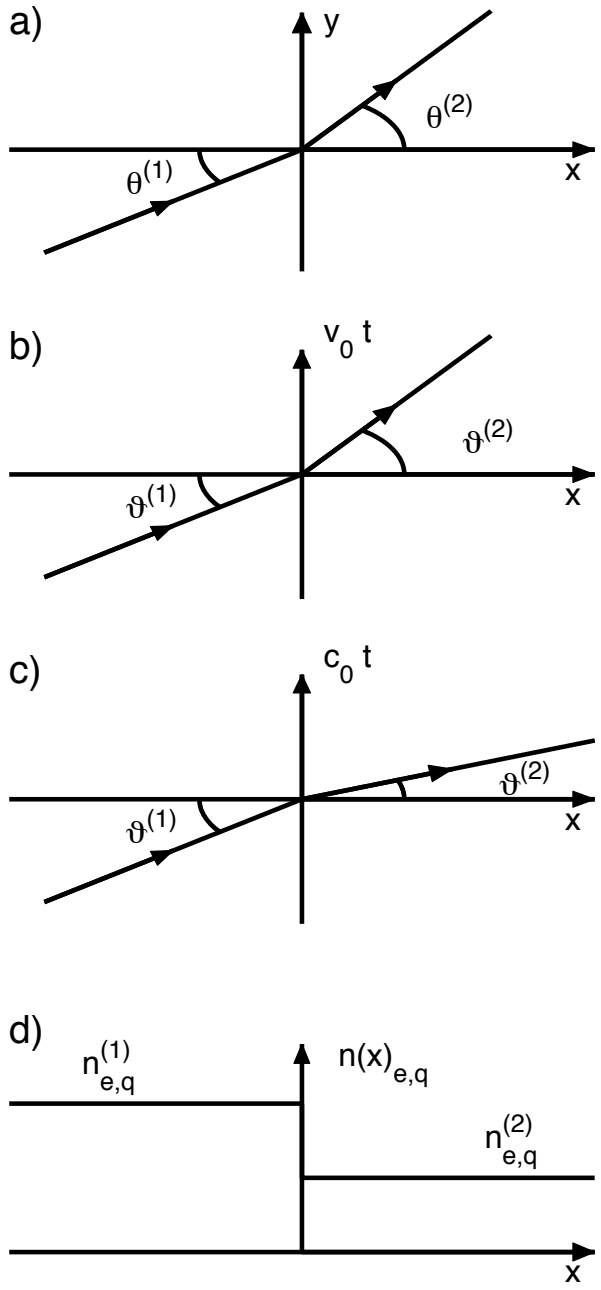

FIG. 1: (a) Path taken by an optical or quantum wave through two adjoining media with different refractive indices $n^{(1)}$ and $n^{(2)}$. (b) Trajectory of a wave packet in $1+1$-dimensional spacetime, with time measured in terms of propagation distance. The wave packet exhibits a refraction-like phenomenon with the refraction angles satisfying a modified law of refraction given by Eq. (12b). (c) Trajectory of an optical wave in $1+1$-dimensional spacetime when crossing between two nondispersive media. The equivalent law describing its refraction in spacetime is different [see Eq. (12a)]. (d) Refractive index as a function of the spatial coordinate.

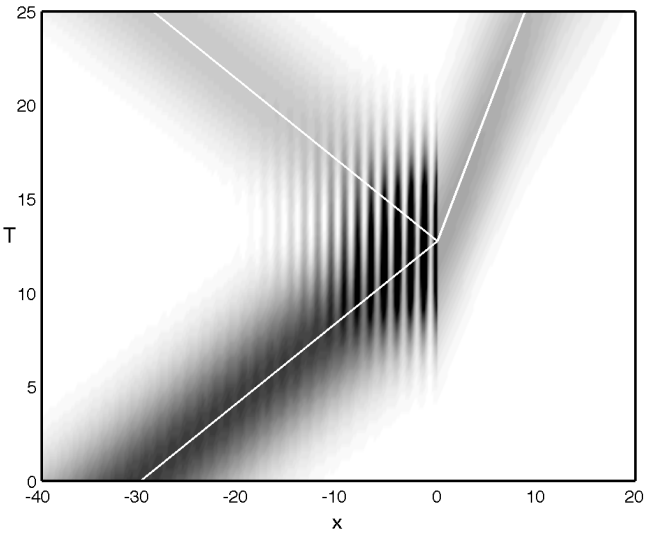

FIG. 2: Refraction in spacetime for a Gaussian wave packet. The coordinate x is measured in terms of an arbitrary unit of length ℓ, and $T = \hbar t/m\ell^2$. The parameters used in the simulation are $\Delta k = 0.1/\ell$, $x_0 = -30\,\ell$, $k_0 = 2.35/\ell$, and $V_0 = 2.5\hbar^2/m\ell^2$. The wave packet is refracted away from the normal as a result of the positive potential step. The white lines are plots of the trajectories for the incident, reflected, and the refracted rays obeying the law of refraction in spacetime [Eq. (12b)].

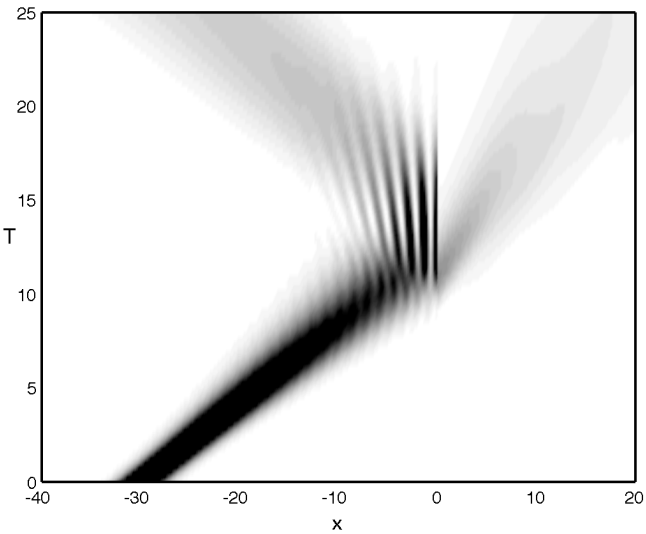

FIG. 3: Refraction in spacetime for a Gaussian wave packet that broadens significantly during the scattering event. The parameters are the same as in Fig. 2, except that $\Delta k = 0.5/\ell$. The interference fringes between the incident and reflected components are seen to be curved, in contrast to the phase fronts in Fig. 2 which are straight.

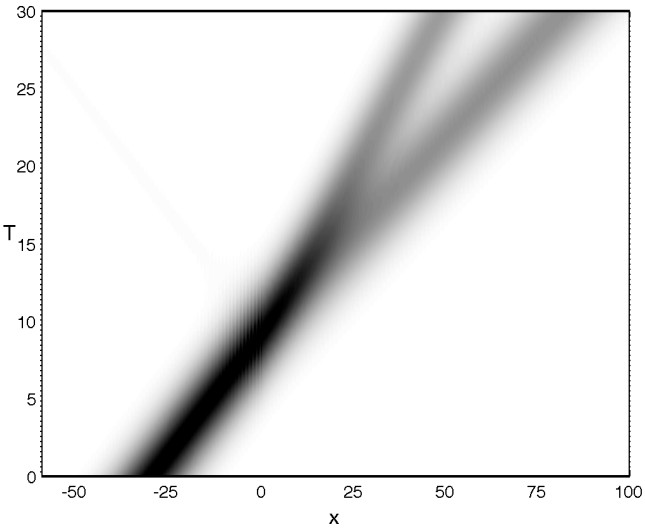

FIG. 4: Birefringence in spacetime of a Gaussian wave packet prepared in a field-free region as a linear superposition of spin-↑ and spin-↓ components with equal weight. The parameters are $\Delta k = 0.1/\ell$, $x_0 = -30\ell$, $k_0 = 3.5/\ell$, and $\mu|B| = 2.5\,\hbar^2/m\ell^2$. The incident wave packet crosses into a region with finite magnetic field at approximately $T = 7.5$ (i.e., $t = 7.5\,m\ell^2/\hbar$). The two distinct paths visible for the transmitted wave correspond to its separated spin-↑ and spin-↓ components.