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Testing for Seasonal Unit Roots when Residuals Contain Serial Correlations under HEGY Test Framework

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Abstract

This paper introduces a corrected test statistic for testing seasonal unit roots when residuals contain serial correlations, based on the HEGY test proposed by Hylleberg, Engle, Granger and Yoo (1990). The serial correlations in the residuals of test regression are accommodated by making corrections to the commonly used HEGY t statistics. The asymptotic distributions of the corrected t statistics are free from nuisance parameters. The size and power properties of the corrected statistics for quarterly and montly data are investigated. Based on our simulations, the corrected statistics for monthly data have more power compared with the commonly used HEGY test statistics, but they also have size distortions when there are strong negative seasonal correlations in the residuals.

Key words: HEGY test; Serial correlations, Corrected t statistics

1 Introduction

Testing seasonal unit roots has been discussed for a long time. Unlike the stationary seasonal time series, the series with seasonal unit roots have changing seasonal patterns and increasing variance, therefore it is important to verify the existence of seasonal unit roots. Several tests have been proposed for testing presence of seasonal unit roots, Dickey, Hasza and Fuller (1984) proposed the DHF test which could tests the existence of seasonal unit roots and non-seasonal unit root as a whole. Hylleberg, Engle, Granger and Yoo (1990) proposed the HEGY tests which could test seasonal unit roots in quarterly data separately. Thus the HEGY test could provide us more information for choosing right differencing filter. Franses (1990), Beaulieu and Miron (1992) extend the HEGY tests to monthly data.

In previous research of HEGY type tests, the serial correlations in residuals of test equation are accommodated by including lags of dependent variable, see Hylleberg, Engle, Granger and Yoo (1990), Beaulieu and Miron (1992). When there are serial correlations in the residuals, the test loses power, see Ghysels, Lee and Noh (1994), Rodrigues and Osborn (1999). In this paper we propose another method for accommodating serial correlations in the residuals of the HEGY test equation using similar technique of commonly used Phillip-Perron unit root test. We first derive the asymptotic distributions of the HEGY t statistics when the residuals are serial correlated, and then derive the correction terms based on the asymptotic distributions. The process above leads to the corrected t statistics whose asymptotic distributions are free from nuisance parameters.

The rest of the paper is organized as follows. The 2nd part introduces seasonal unit roots and the HEGY test procedure for data at any frequency. The 3rd part gives the corrected t statistics and their asymptotic distributions. In the 4th parts we investigate power and size properties of our corrected t statistics for quarterly and monthly data. The 5th part gives conclusions.

2 Seasonal unit roots and HEGY test procedure

2.1 Seasonal unit roots

For a basic autoregressive polynomial $\varphi(B) = 1-B^S$, where B is the lag operator, "S" is the number of observations in a seasonal pattern which repeats regularly, "S" may be even or odd number, e.g., S=4 for quarterly data and S=7 for daily data. The equation $\varphi(z) = 0$ has S roots on the unit circle:

$$z_k = e^{\frac{2\pi i}{S}k} = cos(\frac{2k\pi}{S}) + isin(\frac{2k\pi}{S}), k = 0, 1, 2, ..., (S-1)$$

where i is the imaginary unit. Each root z_k is related to a specific frequency $\frac{2k\pi}{S}$. When $k = 0$, the root z_k is called non-seasonal unit root. The other roots are called seasonal unit roots.

We rearrange the S frequencies by putting the conjugation frequencies together:

a) When S is an even number, the S frequencies are ordered as:

$$\theta_m = \begin{cases} 0 & m = 1 \\ \frac{m-1}{S}\pi & m = 2, 4, ..., (S-2) \\ 2\pi - \frac{m-2}{S}\pi & m = 3, 5, ..., (S-1) \\ \pi & m = S \end{cases} \tag{2.1}$$

In 2.1, θ_m and θ_{m+1} are conjugation frequencies if $m = 2, 4, ..., (S-2)$.

b) When S is an odd number, there is no unit root at frequency π, the S frequencies are ordered as:

$$\theta_m = \begin{cases} 0 & m = 1 \\ \frac{m}{S}\pi & m = 2, 4, ..., (S-1) \\ 2\pi - \frac{m-1}{S}\pi & m = 3, 5, ..., S \end{cases} \tag{2.2}$$

In 2.2, θ_m and θ_{m+1} are conjugation frequencies if $m = 2, 4, ..., (S-1)$.

We make the following notations for simplification. For $m = 1$ we still use $m = 1$. For $m = S$ when S is even, denote $m = \pi$. For the rest, when m is even, i.e., $m = 2, 4, ..., (S-2)$ in 2.1 and

$m = 2, 4, ..., (S-1)$ in 2.2, denote $m = m_{even}$; when m is odd, i.e., $m = 3, 5, ..., (S-1)$ in 2.1 and $m = 3, 5, ..., S$ in 2.2, denote $m = m_{odd}$. The notations are are used throughout the paper.

HEGY procedure: The HEGY test procedure has the advantage of testing existence of seasonal unit roots separately and therefore provides more information for choosing right differencing filter. It is proposed by Hylleberg, Engle, Granger and Yoo (1990) for testing seasonal unit roots in quarterly data. Meng and He (2013) give the HEGY test for data at any frequency. Here we directly give Lemma 1 which introduces the test equation employed for HEGY tests for data at any frequency. For details and proof see Meng and He (2013).

Lemma 1: Consider the seasonal time series $\{y_t : t \in Z_+\}$ with frequency S, assume the series satisfy the autoregressive model 2.3:

$$\varphi(B)y_t = \varepsilon_t \tag{2.3}$$

where $\varphi(B)$ is an autoregressive polynomial with order Q, Q is finite or infinite, assuming Q>S when it is finite, and ε_t is a sequence of $iid(0, \sigma^2)$ random variables, $0 < \sigma^2 < \infty$. The regression model 2.4 is employed for testing seasonal unit roots in $\{y_t\}$:

$$\varphi^*(B)(1 - B^S)y_t = \sum_{m=1}^{s} \rho_m x_{m,t} + \varepsilon_t \tag{2.4}$$

where $\varphi^*(B)$ is an autoregressive polynomial with order Q-S, $x_{m,t} = \zeta_m(B)y_t$,

$$\zeta_m(B) = \begin{cases} \sum_{j=1}^{S} \cos(j * \theta_m)B^j & m = 1, \pi, m_{even} \\ \sum_{j=1}^{S} \sin(j * \theta_{m-1})B^j & m = m_{odd} \end{cases}$$

2.4 is estimated by ordinary least squares method. For unit roots at frequencies 0 and π, unit root exists when the corresponding regressors have 0 parameter. The statistics used are t statistics: $t_m = \hat{\rho}_m / \hat{\sigma}_{\rho m}$, $\hat{\rho}_m$ is the estimate of ρ_m and $\hat{\sigma}_{\rho m}$ is the standard error of $\hat{\rho}_m$. For other conjugation frequencies, only both parameters equal to 0 could verify the existence of unit roots. The F statistics are used. Another option is the using the test statistic $\frac{1}{2}(t_m^2 + t_{m+1}^2)$, $m = m_{even}$, and it is proofed that they have the same asymptotic distributions with the F statistics, see Meng and He (2013).

In regression model 2.4, the series y_t have the property that $E(y_t) = 0$. When there are seasonal means in the series, 2.4 is amended to contain seasonal dummies which leads to regression model 2.5:

$$\varphi^*(B)(1 - B^S)y_t = \sum_{m=1}^{S} \rho_m x_{m,t} + \sum_{i=1}^{S} c_i D_{i,t} + \varepsilon_t \tag{2.5}$$

where $D_{i,t}, i = 1, 2, ..., S$ are seasonal dummy variables which equals to 1 when y_t is at the i^{th} time unit in a seasonal period and 0 elsewhere. The test procedure are the still the same, but the distributions of the t statistics changes.

In regression model 2.4, a common assumption is that serial correlations in residuals could be accommodated by $\varphi^*(B)$. However, when there is serial correlations in the residuals, the order of

$\varphi(B)$ should be infinity. Therefore the order of $\varphi^*(B)$ should also be infinity which could not be achieved practically. In this paper we proposed corrected statistics for dealing with serial correlations in residuals.

3 Corrected test statistics for testing seasonal unit roots

In HEGY procedure, we assume the residuals ε_t in 2.4 and 2.5 are uncorrelated. When this assumption is violated, a common approach for accommodating the serial correlations of the residuals is to include lags of $(1 - B^S)y_t$ in test equation. For previous researches, it is shown that the test loses power when there are serial correlations in the residuals, see Ghysels, Lee and Noh (1994), Rodrigues and Osborn (1999). Also when there are negative seasonal MA components in the series, there are size distortions for test statistics. The shortages could also be seen in our simulation results in Section 4.

When testing unit root in series without seasonality, Phillips (1987) and Phillips and Perron (1988) suggest a method to deal with the serial correlations in residuals. The Phillips-Perron test makes adjustments of the test statistics which would make the statistics have the same asymptotic distributions with the commonly used Dickey-Fuller test. In this section, we follow the same technique, generalize it to HEGY test procedure and propose corrected test statistics for testing seasonal unit roots. First we give the asymptotic distributions of the t statistics when residuals in test regression are serial correlated, then we give our statistics based on these distributions, in the 3rd subsection we give the asymptotic distributions of our corrected t statistics.

3.1 Asymptotic distributions of the t statistics when residuals are serial correlated.

Consider the following model:

$$(1 - L^S)y_t = u_t \tag{3.1}$$

where $u_t = \Psi(B)\varepsilon_t$, $\Psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots$ and ε_t is iid $(0, \sigma^2)$.

Then we have the regression model based on the HEGY decomposition:

$$(1 - B^S)y_t = \sum_{m=1}^{s} \rho_m x_{m,t} + u_t \tag{3.2}$$

where $x_{m,t}$ is the same definition as in Lemma 1. We re-express $u_t = \Psi(B)\varepsilon_t$ into vector forms:

$$U_j = \sum_{r=0}^{\infty} \Phi_r E_{j-r} \tag{3.3}$$

where $U_j = (u_{Sj+1}, u_{Sj+2}, ..., u_{Sj+S})'$, $E_j = (\varepsilon_{Sj+1}, \varepsilon_{Sj+2}, ..., \varepsilon_{Sj+S})'$, and

$$\Phi_0=\begin{bmatrix} \psi_0 & 0 & 0 & \dots & 0 \\ \psi_1 & \psi_0 & 0 & \dots & 0 \\ \psi_2 & \psi_1 & \psi_0 & \dots & 0 \\ \dots & \dots & \dots & & \dots \\ \psi_{S-1} & \psi_{S-2} & \psi_{S-3} & \dots & \psi_0 \end{bmatrix} \Phi_r=\begin{bmatrix} \psi_{Sr} & \psi_{Sr-1} & \psi_{Sr-2} & \dots & \psi_{S(r-1)+1} \\ \psi_{Sr+1} & \psi_{Sr} & \psi_{Sr-1} & \dots & \psi_{S(r-1)+2} \\ \psi_{Sr+2} & \psi_{Sr+1} & \psi_{Sr} & \dots & \psi_{Sr-2} \\ \dots & \dots & \dots & & \dots \\ \psi_{S(r+1)-1} & \psi_{S(r+1)-2} & \psi_{S(r+1)-3} & \dots & \psi_{Sr} \end{bmatrix}$$

With the expression above we could derive the following matrixes:

$\wedge = \sigma \sum_{r=0}^{\infty} \Phi_r,\ \wedge^* = \wedge^T \wedge,\ \Gamma_r = \sigma^2 \sum_{v=0}^{\infty} \Phi_{r+v}\Phi_v^T,\ \Gamma = \sum_{r=1}^{\infty} \Gamma_r$

In order to derive the asymptotic distributions of the t statistics when residuals are serially correlated, the following assumptions are needed.

Assumption 1: ε_t is a sequence of $iid(0,\sigma^2)$ random variables, $0 < \sigma^2 < \infty$.

Assumption 2: $\sum_{j=0}^{\infty} j\,|\psi_j| < \infty$, where "||" denote the absolute value.

With the 2 assumptions above, Theorem 1 gives the the asymptotic distributions of the t statistics when there is no deterministic component in the regression model.

Theorem 1: Consider the regression model 3.2 with assumptions 1-2 fulfilled, under the hypothesis $H_{01} : \rho_1 = 0$, the t statistics t_1 have the asymptotic distributions below:

$$t_1 \xrightarrow{L} \frac{\sum_{j=1}^{S} \wedge_j^* \int_0^1 W_1(r)dW_1(r) + \frac{1}{S}\sum_{p=1}^{S}\sum_{q=1}^{S}\Gamma_{p,q} + \frac{1}{S}\sum_{q=2}^{S}\sum_{p=1}^{q-1}\Gamma_{0;p,q}}{(\gamma_0 \sum_{j=1}^{S} \wedge_j^* \int_0^1 W_1(r)^2 dr)^{1/2}}$$

In the cases S is even, under the hypothesis $H_{0\pi} : \rho_\pi = 0$, the t statistics t_π have the asymptotic distributions below:

$$t_\pi \xrightarrow{L} \frac{\sum_{j=1}^{S}(-1)^{j-1} \wedge_j^* \int_0^1 W_m(r)dW_m(r) + \frac{1}{S}\sum_{p=1}^{S}\sum_{q=1}^{S}(-1)^{p-q}\Gamma_{p,q} + \frac{1}{S}\sum_{q=2}^{S}\sum_{p=1}^{q-1}(-1)^{p-q}\Gamma_{0;p,q}}{[\gamma_0 \sum_{k=1}^{S}(-1)^{j-1} \wedge_j^* \int_0^1 W_2(r)^2 dr]^{1/2}}$$

Under the hypothesis $H_{0m} : \rho_m = \rho_{m+1} = 0$, $m = m_{even}$, the t statistics have the asymptotic distributions below: when $m = m_{even}$,

$$t_m \xrightarrow{L} \frac{\sum_{k=1}^{S} \wedge_j^* cos[(j-1)\theta_m][\int_0^1 W_m(r)dW_m(r) + \int_0^1 W_{m+1}(r)dW_{m+1}(r)]}{\{\gamma_0 \sum_{k=1}^{S} \wedge_j^* cos[(j-1)\theta_m][\int_0^1 W_m(r)^2 dr + \int_0^1 W_{m+1}(r)^2 dr]\}^{1/2}} + \frac{\frac{2}{S}\sum_{p=1}^{S}\sum_{q=1}^{S} cos[\theta_m(p-q)]\Gamma_{p,q} + \frac{2}{S}\sum_{q=2}^{S}\sum_{p=1}^{q-1} cos[\theta_m(p-q)]\Gamma_{0;p,q}}{\{\gamma_0 \sum_{k=1}^{S} \wedge_j^* cos[(j-1)\theta_m][\int_0^1 W_m(r)^2 dr + \int_0^1 W_{m+1}(r)^2 dr]\}^{1/2}}$$

When $m = m_{odd}$

$$t_m \xrightarrow{L} \frac{\sum_{j=1}^{S} \wedge_j^* cos[(j-1)\theta_{m-1}][\int_0^1 W_m(r)dW_{m-1}(r) - \int_0^1 W_{m-1}(r)dW_m(r)]}{\{\gamma_0 \sum_{j=1}^{S} \wedge_j^* cos[(j-1)\theta_{m-1}][\int_0^1 W_{m-1}(r)^2dr + \int_0^1 W_m(r)^2dr]\}^{1/2}} + \frac{\frac{2}{S}\sum_{p=1}^{S}\sum_{q=1}^{S} sin[\theta_{m-1}(p-q)]\Gamma_{p,q} + \frac{2}{S}\sum_{q=2}^{S}\sum_{p=1}^{q-1} sin[\theta_{m-1}(p-q)]\Gamma_{0;p,q}}{\{\gamma_0 \sum_{j=1}^{S} \wedge_j^* cos[(j-1)\theta_{m-1}][\int_0^1 W_{m-1}(r)^2dr + \int_0^1 W_m(r)^2dr]\}^{1/2}}$$

where W_m are mutually independent standard Brownian motions, $\gamma_0 = \sigma^2 \sum_{v=0}^{\infty} \psi_v^2$, $\wedge_j^*, j = 1, 2, .., S$ are the j^{th} element of the first row of $\wedge^*$. $\Gamma_{p,q}$ is the row p, column q element in Γ, and $\Gamma_{0;p,q}$ is row p, column q element in Γ_0. For details of proof, see Appendix 1.1.

We can see, when there are serial correlations in residuals u_t, the asymptotic distributions of the t statistics depend on the parameters ψ_i. The asymptotic distributions for $t_m, m = m_{even}$ are not the same since they depend θ_m. The situation is the same for the distributions of $t_m, m = m_{odd}$.

When the series have seasonal means, seasonal dummies are included in the regression model, the model becomes:

$$(1 - B^S)y_t = \sum_{m=1}^{S} \rho_m x_{m,t} + \sum_{i=1}^{S} c_i D_{i,t} + u_t \tag{3.4}$$

$D_{i,t}$ is defined same as in 2.5. Theorem 2 gives the asymptotic distributions of the t statistics when 3.4 is employed for test.

Theorem 2: Consider the regression model 3.4 with assumptions 1-2 fulfilled, under the hypothesis $H_{01} : \rho_1 = 0$, the t statistics t_1 have the asymptotic distributions below:

$$t_1 \xrightarrow{L} \frac{\sum_{j=1}^{S} \wedge_j^* [\int_0^1 W_1(r)dW_1(r) - W_1(1)\int_0^1 W_1(r)dr] + \sum_{p=1}^{S}\sum_{q=1}^{S}\Gamma_{p,q} + \sum_{q=2}^{S}\sum_{p=1}^{q-1}\Gamma_{0;p,q}}{\{\gamma_0 \sum_{j=1}^{S} \wedge_j^* [\int_0^1 W_1(r)^2dr + (\int_0^1 W_1(r)dr)^2]\}^{1/2}}$$

In the cases S is even, under the hypothesis $H_{0\pi} : \rho_\pi = 0$, the t statistics t_π have the asymptotic distributions below:

$$t_m \xrightarrow{L} \frac{\sum_{j=1}^{S}(-1)^{j-1} \wedge_j^* [\int_0^1 W_m(r)dW_m(r) - W_m(1)\int_0^1 W_m(r)dr]}{\{\gamma_0 \sum_{k=1}^{S}(-1)^{j-1} \wedge_j^* [\int_0^1 W_m(r)^2dr - (\int_0^1 W_m(r)dr)^2\}^{1/2}} + \frac{\sum_{p=1}^{S}\sum_{q=1}^{S}(-1)^{p-q}\Gamma_{p,q} + \sum_{q=2}^{S}\sum_{p=1}^{q-1}(-1)^{p-q}\Gamma_{0;p,q}}{\{\gamma_0 \sum_{j=1}^{S}(-1)^{j-1} \wedge_j^* [\int_0^1 W_m(r)^2dr - (\int_0^1 W_m(r)dr)^2\}^{1/2}}$$

Under the hypothesis $H_{0m} : \rho_m = \rho_{m+1} = 0$, $m = m_{even}$, the t statistics have the asymptotic distributions below: when $m = m_{even}$,

$$t_m \overset{L}{\to} \frac{\sum_{j=1}^{S} \wedge_j^* cos[(j-1)\theta_m][\int_0^1 W_m(r)dW_m(r) + \int_0^1 W_{m+1}(r)dW_{m+1}(r) + W_{cos}^*]}{\{\gamma_0 \sum_{j=1}^{S} \wedge_j^* cos[(j-1)\theta_m][\int_0^1 W_m(r)^2dr + \int_0^1 W_{m+1}(r)^2dr + W_{cos}^{**}]\}^{1/2}} +$$

$$\frac{\sum_{p=1}^{S}\sum_{q=1}^{S} cos[\theta_m(p-q)]\Gamma_{p,q} + \sum_{q=2}^{S}\sum_{p=1}^{q-1} cos[\theta_m(p-q)]\Gamma_{0;p,q}}{\{\gamma_0 \sum_{j=1}^{S} \wedge_k^* cos[(j-1)\theta_m][\int_0^1 W_m(r)^2dr + \int_0^1 W_{m+1}(r)^2dr] + W_{cos}^{**}\}^{1/2}}$$

where $W_{cos}^* = -W_m(1)\int_0^1 W_m(r)dr - W_{m+1}(1)\int_0^1 W_{m+1}(r)$,

$W_{cos}^{**} = -(\int_0^1 W_m(r)dr)^2 - (\int_0^1 W_{m+1}(r)dr)^2$

When $m = m_{odd}$:

$$t_m \overset{L}{\to} \frac{\sum_{j=1}^{S} \wedge_j^* cos[(j-1)\theta_{m-1}][\int_0^1 W_m(r)dW_{m-1}(r) - \int_0^1 W_{m-1}(r)dW_m(r) + W_{sin}^*]}{\{\gamma_0 \sum_{j=1}^{S} \wedge_j^* cos[(j-1)\theta_{m-1}][\int_0^1 W_{m-1}(r)^2dr + \int_0^1 W_m(r)^2dr + W_{sin}^{**}]\}^{1/2}} +$$

$$\frac{\sum_{p=1}^{S}\sum_{q=1}^{S} sin[\theta_{m-1}(p-q)]\Gamma_{p,q} + \sum_{q=2}^{S}\sum_{p=1}^{q-1} sin[\theta_{m-1}(p-q)]\Gamma_{0;p,q}}{\{\gamma_0 \sum_{j=1}^{S} \wedge_j^* cos[(j-1)\theta_{m-1}][\int_0^1 W_{m-1}(r)^2dr + \int_0^1 W_m(r)^2dr] + W_{sin}^{**}\}^{1/2}}$$

where $W_{sin}^* = -W_m(1)\int_0^1 W_{m-1}(r)dr - W_{m-1}(1)\int_0^1 W_m(r)dr$, $W_{sin}^{**} = -(\int_0^1 W_{m-1}(r)dr)^2 - (\int_0^1 W_m(r)dr)^2$

For details of proof, see Appendix 1.2.

3.2 Corrected test statistics

The asymptotic distributions in Theorem 1 and 2 contain nuisance parameters. We propose our corrected t statistics for testing seasonal unit roots which make corrections of the t statistics. The corrected statistics have asymptotic distributions free from nuisance parameters. The statistics are given below:

Corrected test statistics: With the vector expressions, we define the following t statistics to test parameters in regression model 3.2:

For $m = 1, \pi, m_{even}$,

$$\widetilde{t}_m = G_m\{t_m - \frac{T\hat{\sigma}_{\rho_m}}{Ss_T^2}[\sum_{p=1}^{S}\sum_{q=1}^{S} cos[\theta_m(p-q)]\hat{\Gamma}_{p,q} + \sum_{q=2}^{S}\sum_{p=1}^{q-1} cos[\theta_m(p-q)]\hat{\Gamma}_{0;p,q}]\}$$

For $m = m_{odd}$:

$$\widetilde{t}_m = G_{m-1}\{t_m - \frac{T\hat{\sigma}_{\rho_m}}{Ss_T^2}[\sum_{p=1}^{S}\sum_{q=1}^{S} sin[\theta_{m-1}(p-q)]\hat{\Gamma}_{p,q} + \sum_{q=2}^{S}\sum_{p=1}^{q-1} sin[\theta_{m-1}(p-q)]\hat{\Gamma}_{0;p,q}]\}$$

where $G_m = \{\frac{s_T^2}{\sum_{j=1}^{S} \widetilde{\wedge}_j^* cos[(j-1)\theta_m]}\}^{\frac{1}{2}}$, s_T^2 is the estimator of variance of u_t, $\hat{\Gamma}_{p,q}$ is the estimator of $\Gamma_{p,q}$,

$\hat{\Gamma}_{0;p,q}$ is the estimator of $\Gamma_{0;p,q}$, $\widetilde{\wedge}_j^*$ is the estimator of $\wedge_j^*$.

In practical issue, we estimate regression model 3.2 by ordinary least squares, and get the t statistics introduced in Lemma 1. Then we estimate the model $\hat{u}_t = \Psi(B)\varepsilon_t$ with the residuals in 3.2, deriving $\hat{\psi}_i, i = 0, 1, 2, ...l$ and $\hat{\sigma}$ which are estimators of $\psi_i, i = 0, 1, 2, ...$ and σ. For the order l, a preliminary investigation of the sample autocorrelations of $u_t = y_t - y_{t-S}$ would help us choose an appropriate order, following the suggestions given by Phillips (1987). The maximum likelihood method is used to estimate the parameters. With $\hat{\psi}_i$ we could get the estimator of $\Phi_i, i = 0, 1, 2, ...$. Moreover, with $\hat{\sigma}$, we could derive $\hat{\Gamma}$, $\hat{\Gamma}_0$ and $\widetilde{\wedge}^*$ by basic matrix operation introduced in Subsection 3.1.

Similar with the HEGY procedure, the parameter ρ_1 corresponds to unit roots at frequencies 0, and when S is even, ρ_2 corresponds to unit roots at frequencies π. The null and alternative hypothesis are

$$H_{0m} : \rho_m = 0. \qquad H_{am} : \rho_m < 0 (m = 1, \pi).$$

$\widetilde{t}_m, m = 1, \pi$ is used to test if 2 parameters equal to 0. If the null hypothesis is rejected, the series is stationary at that frequency. For other unit roots, the parameters appear in pairs corresponding with conjugation frequencies. Both parameters are zero could verify the existence of unit roots. The null and alternative hypothesis are given below:

$$H_{0m} : \rho_m = \rho_{m+1} = 0. \qquad H_{am} : \rho_m \neq 0 \; or \; \rho_{m+1} \neq 0.$$

where $m = m_{even}$. The statistics $\frac{1}{2}(\widetilde{t}_m^2 + \widetilde{t}_{m+1}^2)$ are used, denoted as $\widetilde{F}_{m,}$. If the null hypothesis is not rejected, the unit roots exist at corresponding 2 frequencies, and if the null hypothesis is rejected, the series is stationary at both frequencies. The second strategy is testing 2 parameters separately. Test $H^*_{0(m+1)} : \rho_{m+1} = 0$ against $H^*_{a(m+1)} : \rho_{m+1} \neq 0$ by t statistics $\widetilde{t}_{m+1}$ at the beginning, where $m = m_{even}$. If the null hypothesis is not rejected, then examine $H^*_{0m} : \rho_m = 0$ against $H^*_{am} : \rho_m < 0$ by t statistics $\widetilde{t}_m$. If the null hypothesis is not rejected again, there are unit roots at the corresponding 2 frequencies.

The corrected t statistics could also be used to test the presence of all the S unit root as a whole with a joint test for all the S parameters. The null hypothesis is $H_0 : \rho_1 = \rho_2 = ... = \rho_S = 0$ while the alternative is H_a: The series is seasonally stationary. The statistics used is $\widetilde{F}_{all} = \frac{1}{S}\sum_{m=1}^{S} \widetilde{t}_m^2$. If the null hypothesis is accepted, $(1 - B^S)$ is needed to render the series stationary. If the null hypothesis is rejected, the series is seasonally stationary.

There is also the case when seasonal dummies are included in the regression 3.4. When 3.4 is employed, our corrected t statistics could also be used for testing seasonal unit roots. The procedure is still the same as above, but the distributions of the t statistics change. More discussions are given in subsection 3.3.

3.3 Asymptotic distribution of corrected test statistics.

This subsection gives the asymptotic distributions of the corrected t statistics in previous subsection. Theorem 3 gives the asymptotic distributions of the corrected t statistics when test regression 3.2

is employed for the test. Theorem 4 gives the asymptotic distributions of the corrected t statistics when test regression 3.4 is employed for test. These asymptotic distributions are free from nuisance parameters. Note that when there are no serial correlations in the residuals in 3.2 and 3.4, $\Psi(B) = 1$, and it lead to the result that Φ_0 is identity matrix and $\Phi_i = 0, i = 1, 2, ...$, and we could get that the asymptotic distributions of the uncorrected t statistics are the same as those of the corrected t statistics.

Theorem 3: Consider the regression model 3.2 with assumptions 1-2 fulfilled, under the hypothesis $H_{0m} : \rho_m = 0, m = 1, \pi$, the corrected t statistics $\widetilde{t}_m$ have the asymptotic distributions below:

$$\widetilde{t}_m \xrightarrow{L} \frac{\int_0^1 W_m(r) dW_m(r)}{(\int_0^1 W_m(r)^2 dr)^{1/2}}$$

Under the hypothesis $H_{0m} : \rho_m = \rho_{m+1} = 0,\ m = m_{even}$, the t statistics have the asymptotic distributions below:

$$\widetilde{t}_m \xrightarrow{L} \begin{cases} \frac{\int_0^1 W_m(r) dW_m(r) + \int_0^1 W_{m+1}(r) dW_{m+1}(r)}{[\int_0^1 W_m(r)^2 dr + \int_0^1 W_{m+1}(r)^2 dr]^{1/2}} & m = m_{even} \\ \frac{\int_0^1 W_m(r) dW_{m-1}(r) - \int_0^1 W_{m-1}(r) dW_m(r)}{[\int_0^1 W_{m-1}(r)^2 dr + \int_0^1 W_m(r)^2 dr]^{1/2}} & m = m_{odd} \end{cases}$$

" $\xrightarrow{L}$ " stands for converge in distribution, $W_i, i = 1, 2, ..., S$ are mutually independent standard Brownian motions. For details of proof see Appendix 1.3.

The distributions are the same with those of the uncorrected t statistics in Meng and He (2012). We could see that, same with the uncorrected t statistics, the distributions of the corrected t statistics when $m = m_{even}$ and $m = m_{odd}$ are the same across frequencies.

When regression model 3.4 is employed for test, the asymptotic distributions of the corrected t statistics change, and we have the following theorem:

Theorem 4. Consider the regression model 3.4 with assumptions 1-2 fulfilled, then under the hypothesis $H_0 : \rho_m = 0, m = 1, \pi$, the t statistics $\widetilde{t}_m$ have the asymptotic distributions below:

$$\widetilde{t}_m \xrightarrow{L} \frac{\int_0^1 W_m(r) dW_m(r) - W_m(1) \int_0^1 W_m(r) dr}{[\int_0^1 W_m(r)^2 dr - (\int_0^1 W_m(r) dr)^2]^{1/2}}$$

Under the hypothesis $H_{0m} : \rho_m = \rho_{m+1} = 0,\ m = m_{even}$, the t statistics have the asymptotic distributions below:

$$\widetilde{t}_m \xrightarrow{L} \begin{cases} \frac{\int_0^1 W_m(r) dW_m(r) + \int_0^1 W_{m+1}(r) dW_{m+1}(r) + W^*_{cos}}{[\int_0^1 W_m(r)^2 dr + \int_0^1 W_{m+1}(r)^2 dr + W^{**}_{cos}]^{1/2}} & m = m_{even} \\ \frac{\int_0^1 W_m(r) dW_{m-1}(r) - \int_0^1 W_{m-1}(r) dW_m(r) + W^*_{sin}}{[\int_0^1 W_m(r)^2 dr + \int_0^1 W_{m-1}(r)^2 dr + W^{**}_{sin}]^{1/2}} & m = m_{odd} \end{cases}$$

where W^*_{cos}, W^{**}_{cos}, W^*_{sin}, W^{**}_{sin} are the same with Theorem 2. For details of the proof, see Appendix 1.3.

We could see that the inclusion of seasonal dummies would affect the distributions of the corrected t statistics. The corrected t statistics for $m = m_{even}$ again have the same asymptotic distributions, and the corrected t statistics for $m = m_{odd}$ also have the same asymptotic distributions.

4 Monte-Carlo Simulation

In this part we give the size and power analysis of the uncorrected and corrected t statistics. A Monte-Carlo simulation method is used to provide the size and power estimates of our corrected t statistics. We focus on the case when testing seasonal unit roots for quarterly data and monthly data, i.e., S=4 and S=12. We compare the size and power properties of the corrected t statistics with those of the uncorrected ones. First the simulation procedure for S=4 is introduced, and the procedure for S=12 are similar with that.

i) For quarterly data, the following data generating processes (DGP) are used:

$DGP1: \ (1 - B^4)y_t = \varepsilon_t + \psi_1\varepsilon_{t-1}$

$DGP2: \ (1 - B^4)y_t = \varepsilon_t + \psi_4\varepsilon_{t-4}$

$DGP3: \ (1 - 0.9B^4)y_t = \varepsilon_t + \psi_1\varepsilon_{t-1}$

$DGP4: \ (1 - 0.9B^4)y_t = \varepsilon_t + \psi_4\varepsilon_{t-4}$

where $\varepsilon_t \sim N(0, 1)$.

For size analysis, we simulate data from DGP1 and DGP2 with sample size set to 480. ψ_1 and ψ_4 are chosen to be ± 0.1 ± 0.5 and ± 0.9. For the uncorrected statistics, we estimate regression model 2.4, the order of lag augmentations $\varphi^*(B)$ is chosen based on AIC criteria. Then we derive the t statistics with F statistics which are introduced in Subsection 2.1. The test size are based on 5% level. The procedure is repeated for 5000 times, yielding the estimates in Table 1. For the size estimates of our corrected test statistics, the sample sizes and level of test size are the same as above. We estimate regression model 2.4 and get the residuals as well as uncorrected t statistics for the 4 regressors. To get the estimators of $\Psi(B)$ and σ^2 in (3.1), for DGP1, the residuals in last step are modeled by MA(1), and for DGP2, the residuals in last step are modeled by Seasonal Moving Average model order 1 for seasonal part and order 0 for non-seasonal part. The corrected t statistics are compared with the same critical values as the uncorrected ones. The procedure is also repeated for 5000 times, giving the size estimates in Table 2.

For power analysis, we analyze the size-corrected power of the statstics. The procedures are mostly the same as above except DGP1 and DGP2 are changed to DGP3 and DGP4, and the critical values are adjusted by size. The sample size is 480 and test sizes are based on 5% nominal level. The power estimates for uncorrected statistics are given in Table 5, and those for corrected statistics are given in Table 6.

ii) For monthly data, the following data generating processes are used:

$DGP5: \ (1 - B^{12})y_t = \varepsilon_t + \psi_1\varepsilon_{t-1}$

$DGP6: \ (1 - B^{12})y_t = \varepsilon_t + \psi_{12}\varepsilon_{t-12}$

$DGP7: \ (1 - 0.9B^{12})y_t = \varepsilon_t + \psi_1\varepsilon_{t-1}$

$DGP8: \ (1 - 0.9B^{12})y_t = \varepsilon_t + \psi_{12}\varepsilon_{t-12}$

where $\varepsilon_t \sim N(0,1)$.

The simulation procedures for size and power analysis are similar as those in quarterly data. For size analysis, we simulate data from DGP5 and DGP6 with sample size set to 480. For uncorrected statitstics, we estimate model 2.4, the order of $\varphi^*(B)$ is set to 12. The t statistics and F statistics are also derived in the way introduced in Subsection 2.1. The nominal level is also set to 5%. The procedure is repeated for 5000 times, yielding the estimates in Table 3. For the corrected t statistics, we estimate regression model 2.4 and get the residuals as well as uncorrected t statistics for the 12 regressors. To get the estimators of $\Psi(B)$ in (3.1), for DGP5, the residuals in last step are modeled by MA(1). and for DGP6, by Seasonal Moving Average model with order 1 for seasonal part and order 0 for non-seasonal part. The corrected t statistics are compared with the same critical values above. The procedure is also repeated for 5000 times, giving the size estimates in Table 4. For conjugation frequencies, we only provide the estimates for joint statistics.

We estimate the size-adjusted power of our test and HEGY test. DGP7 and DGP8 are used to generate data. The estimation procedures are the same as size analysis except the critical values are size-adjusted. The power estimates for uncorrected statistics are given in Table 7, and those for corrected statistics are given in Table 8.

4.1 Size of Corrected t statistics.

For quarterly data, Table 1 gives the size estimates for the HEGY t statistics and F statistic and Table 2 gives those estimates for correct test statistics. We can see uncorrected statistics have better the size properties. No serious size distortions exist. Especially when there are strong negtive moving average component in the series, i.e., $\psi_4 = -0.9$, the sizes are around 0.15. However the corrected statistics have serious size distortions in DGP2 when $\psi_4 = -0.9$. Also, there are serious size distortions in DGP1 for $t_4$when $\psi_1 = 0.9$, and for t_1 when $\psi_1 = -0.9$. The reason for the size distortion is that the distributions of the corrected t statistics converge quite slowly. For monthly data, Table 3 gives the size estimates for uncorrected statistics and Table 4 gives those for corrected statistics. We have similar conclusion, the uncorrected statistics still have the better performance than the corrected ones when the residuals have strong negative seasonality.

Table 1: Empirical size of uncorrected t statistics at of 5% level. Regression model (2.2) employed. Sample size 480, 5000 replications.

DGP 1				DGP 2			
ψ_1	t_1	t_4	F_2	ψ_4	t_1	t_4	F_2
0.1	0.06	0.06	0.08	0.1	0.06	0.06	0.07
0.5	0.06	0.06	0.07	0.5	0.06	0.06	0.08
0.9	0.06	0.07	0.08	0.9	0.06	0.06	0.08
-0.1	0.06	0.06	0.08	-0.1	0.06	0.06	0.08
-0.5	0.06	0.06	0.07	-0.5	0.06	0.06	0.08
-0.9	0.07	0.07	0.08	-0.9	0.15	0.17	0.13

Table 2: Empirical size of corrected t statistics for quarterly data at of 5% level. Regression model (2.2) employed. Sample size 480, 5000 replications.

DGP 1				DGP 2			
ψ_1	$\widetilde{t}_1$	$\widetilde{t}_4$	$\widetilde{F}_2$	ψ_4	t_1	t_4	$\widetilde{F}_2$
0.1	0.05	0.05	0.06	0.1	0.05	0.05	0.05
0.5	0.05	0.15	0.19	0.5	0.05	0.05	0.05
0.9	0.03	0.99	0.15	0.9	0.05	0.05	0.05
-0.1	0.05	0.05	0.07	-0.1	0.05	0.05	0.05
-0.5	0.16	0.05	0.18	-0.5	0.18	0.19	0.18
-0.9	0.99	0.03	0.15	-0.9	1.00	1.00	1.00

Table 3: Empirical size of uncorrected t statistics for monthly data at of 5% level. AIC Regression model (2.2) employed.Sample size 480, 5000 replications.

DGP5							
ψ_1	t_1	t_{12}	F_2	F_4	F_6	F_8	F_{10}
0.1	0.07	0.07	0.07	0.08	0.07	0.07	0.07
0.5	0.07	0.07	0.08	0.07	0.07	0.07	0.07
0.9	0.06	0.8	0.07	0.07	0.07	0.08	0.07
-0.1	0.05	0.07	0.08	0.07	0.07	0.07	0.07
-0.5	0.06	0.07	0.08	0.08	0.07	0.08	0.07
-0.9	0.8	0.07	0.07	0.07	0.07	0.07	0.07
DGP6							
ψ_{12}	t_1	t_{12}	F_2	F_4	F_6	F_8	F_{10}
0.1	0.06	0.06	0.08	0.07	0.08	0.08	0.08
0.5	0.06	0.06	0.08	0.08	0.08	0.07	0.07
0.9	0.07	0.08	0.07	0.07	0.08	0.07	0.08
-0.1	0.07	0.07	0.07	0.07	0.08	0.07	0.08
-0.5	0.07	0.07	0.07	0.07	0.07	0.06	0.07
-0.9	0.40	0.43	0.41	0.41	0.41	0.39	0.41

Table 4: Empirical size of corrected t statistics for monthly data at of 5% level. Regression model (2.2) employed.Sample size 480, 5000 replications.

DGP5							
ψ_1	$\widetilde{t}_1$	$\widetilde{t}_{12}$	$\widetilde{F}_2$	$\widetilde{F}_4$	$\widetilde{F}_6$	$\widetilde{F}_8$	$\widetilde{F}_{10}$
0.1	0.05	0.05	0.05	0.05	0.05	0.04	0.04
0.5	0.05	0.07	0.05	0.05	0.04	0.04	0.04
0.9	0.03	0.97	0.05	0.05	0.05	0.05	0.06
-0.1	0.05	0.05	0.04	0.05	0.05	0.05	0.05
-0.5	0.07	0.04	0.03	0.04	0.04	0.04	0.05
-0.9	0.92	0.03	0.16	0.06	0.05	0.05	0.05
DGP6							
ψ_{12}	$\widetilde{t}_1$	$\widetilde{t}_{12}$	$\widetilde{F}_2$	$\widetilde{F}_4$	$\widetilde{F}_6$	$\widetilde{F}_8$	$\widetilde{F}_{10}$
0.1	0.05	0.05	0.05	0.05	0.05	0.05	0.05
0.5	0.06	0.06	0.05	0.05	0.04	0.05	0.04
0.9	0.08	0.08	0.04	0.04	0.03	0.04	0.04
-0.1	0.05	0.05	0.05	0.05	0.05	0.05	0.05
-0.5	0.17	0.16	0.16	0.16	0.15	0.16	0.16
-0.9	1.00	1.00	1.00	1.00	1.00	1.00	1.00

4.2 Power of the corrected t statistics

Table 5 and 6 gives the power estimates of the uncorrected and corrected t statistics and F statistics for quarterly data. We can see that for DGP3, the uncorrected statistics have power around 0.5 while the corrected statistics have power around 0.9. The superiority of the corrected statistics is the same for DGP4, where all the corrected statistics have power greater than 0.8.

For monthly data, Table 7 shows that the the power estimates for uncorrected statistics are quite low which are around 0.2. In DGP8 when $\psi_{12} = -0.9$, the power is higher, but still less than 0.5. The power of t statistics and F statistics do not vary a lot. For the corrected statistics, Table 8 shows that the power estimates are larger than those of uncorrected ones. Only $\widetilde{t}_{12}$ when $\psi_1 = 0.9$ and $\widetilde{t}_1$ when $\psi_1 = -0.9$ have smaller power. The power of different statistics vary a little. For DGP8, the superiority of corrected statistics become larger, while all statistics have similar power performance.

Table 5: Empirical power of uncorrected t statistics at 5% level with SARIMA models. Sample size 480, 5000 replications.

DGP 3				DGP 4			
ψ_1	t_1	t_4	F_2	ψ_4	t_1	t_4	F_2
0.1	0.54	0.53	0.63	0.1	0.50	0.53	0.66
0.5	0.48	0.52	0.64	0.5	0.51	0.50	0.60
0.9	0.50	0.55	0.62	0.9	0.44	0.52	0.51
-0.1	0.56	0.50	0.65	-0.1	0.51	0.51	0.65
-0.5	0.54	0.54	0.67	-0.5	0.55	0.58	0.66
-0.9	0.56	0.46	0.64	-0.9	0.69	0.67	0.93

Table 6: Empirical power of corrected t statistics at 5% level with SARIMA models. Sample size 480, 5000 replications.

DGP 3				DGP 4			
ψ_1	$\widetilde{t}_1$	$\widetilde{t}_4$	$\widetilde{F}_2$	ψ_4	$\widetilde{t}_1$	$\widetilde{t}_4$	$\widetilde{F}_2$
0.1	0.90	0.90	0.99	0.1	0.86	0.84	0.98
0.5	0.81	0.88	0.95	0.5	0.85	0.86	0.99
0.9	0.94	0.75	0.84	0.9	0.83	0.84	0.98
-0.1	0.91	0.89	0.99	-0.1	0.92	0.92	0.99
-0.5	0.89	0.83	0.93	-0.5	1.00	1.00	1.00
-0.9	0.74	0.93	0.83	-0.9	0.98	0.98	1.00

5 Conclusion

In this paper we provide another method for accommodating the serial correlations in the residual in HEGY-type test. The traditional way is to include lag augmentations of the dependent variable.

Table 7: Empirical power of uncorrected t statistics for monthly data at of 5% level. Regression model (3.2) employed.Sample size 480, 5000 replications.

DGP7							
ψ_1	t_1	t_{12}	F_2	F_4	F_6	F_8	F_{10}
0.1	0.20	0.18	0.23	0.20	0.21	0.20	0.22
0.5	0.20	0.18	0.22	0.23	0.23	0.22	0.23
0.9	0.20	0.22	0.21	0.20	0.23	0.20	0.23
-0.1	0.22	0.20	0.19	0.22	0.24	0.22	0.20
-0.5	0.20	0.18	0.22	0.20	0.20	0.21	0.21
-0.9	0.19	0.20	0.23	0.22	0.21	0.23	0.21
DGP8							
ψ_{12}	t_1	t_{12}	F_2	F_4	F_6	F_8	F_{10}
0.1	0.20	0.21	0.21	0.20	0.19	0.19	0.19
0.5	0.19	0.21	0.17	0.17	0.17	0.17	0.18
0.9	0.10	0.09	0.23	0.22	0.21	0.22	0.20
-0.1	0.19	0.21	0.22	0.22	0.21	0.22	0.22
-0.5	0.22	0.21	0.28	0.29	0.30	0.29	0.29
-0.9	0.36	0.34	0.47	0.47	0.48	0.49	0.47

Table 8: Empirical power of corrected t statistics for monthly data at of 5% level. Regression model (3.2) employed.Sample size 480, 5000 replications.

DGP7							
ψ_1	$\widetilde{t}_1$	$\widetilde{t}_{12}$	$\widetilde{F}_2$	$\widetilde{F}_4$	$\widetilde{F}_6$	$\widetilde{F}_8$	$\widetilde{F}_{10}$
0.1	0.24	0.26	0.35	0.35	0.38	0.36	0.38
0.5	0.21	0.37	0.31	0.36	0.44	0.53	0.60
0.9	0.26	0.02	0.42	0.57	0.63	0.69	0.62
-0.1	0.27	0.26	0.39	0.36	0.39	0.36	0.35
-0.5	0.31	0.21	0.59	0.52	0.43	0.39	0.34
-0.9	0.09	0.27	0.37	0.53	0.43	0.32	0.28
DGP8							
ψ_{12}	$\widetilde{t}_1$	$\widetilde{t}_{12}$	$\widetilde{F}_2$	$\widetilde{F}_4$	$\widetilde{F}_6$	$\widetilde{F}_8$	$\widetilde{F}_{10}$
0.1	0.24	0.25	0.34	0.32	0.31	0.32	0.32
0.5	0.24	0.24	0.39	0.37	0.40	0.40	0.38
0.9	0.22	0.20	0.48	0.47	0.46	0.47	0.51
-0.1	0.29	0.28	0.43	0.41	0.40	0.41	0.40
-0.5	0.74	0.74	0.89	0.88	0.90	0.89	0.90
-0.9	0.49	0.50	0.70	0.72	0.72	0.71	0.71

Here we use the similar technique to accommodate serial correlation as the commonly used Phillip-Perron statistics for testing unit root, and derive the corrected t statistics for testing seasonal unit roots under the HEGY framework. Our t statistics are derived by making corrections of the HEGY t statistics, and the corrected t statistics have asymptotic distributions without nuisance parameters. The power and size properties of the corrected statistics for quarterly and monthly data are investigated by Monte-Carlo simulations. In comparison with the HEGY statistics, the corrected statistics have size distortions when there are strong negative seasonal moving average component in the series, while the uncorrected ones do not have. But our corrected test statistics have more power than the uncorrected ones, especially when there are seasonality in the residuals.

6 Appendix I

Appendix 1.1

Proof of Theorem 1: (I) Derive the asymptotic distributions.

With the vector form in subsection 3.1, we also get:

$$\Xi_j = \begin{bmatrix} \xi_{j,1} \\ \xi_{j,2} \\ \ldots \\ \xi_{j,S} \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{j} u_{Si+1} \\ \sum_{i=0}^{j} u_{Si+2} \\ \ldots \\ \sum_{i=0}^{j} u_{Si+S} \end{bmatrix}$$

The correlation matrix for U_j is $\Omega = \sigma^2 I$, where I is identity matrix. Let $\Omega = PP'$ denote the Cholesky factorization of Ω. It is easy to see that $P = \sigma I$ under assumption 1. Thus we have $\Phi(1)P = \sigma\Phi(1) = \wedge$.

We directly give the following convergences which could be found on Hamilton (See Hamilton, 1994, Page 548):

$$J^{-1}\sum_{j=1}^{J} \Xi_{j-1}U_j' \xrightarrow{L} \wedge \int_0^1 B(r)[dB(r)]' \wedge' + \Gamma \tag{6.1}$$

$$J^{-1}\sum_{j=1}^{J} U_j U_{j-r}' \xrightarrow{p} \Gamma_r \, r = 0, 1, 2, \ldots \tag{6.2}$$

$$J^{-2}\sum_{j=1}^{J} \Xi_{j-1}\Xi_{j-1}' \xrightarrow{L} \wedge \int_0^1 B(r)[dB(r)]' \wedge' \tag{6.3}$$

The t statistics are derived by $t_m = \frac{\hat{\rho}_m}{\hat{\sigma}_{\rho_m}}$, where $\hat{\rho}_m$ is the estimate of ρ_m and $\hat{\sigma}_{\rho_m}$ is the standard error of $\hat{\rho}_m$. The numerator:$\rho = [\rho_1, \ldots, \rho_S]' = (X'X)^{-1}X'u$, where $X = [x_{1,t}, x_{2,t}, \ldots, x_{s,t}]$. Consider the diagonal elements in $X'X$ first, when $m = 1, \pi, m_{even}$:

$$
\begin{aligned}
J^{-2}\sum_{t=0}^{T} x_{m,t}^2 &= J^{-2}\sum_{j=0}^{J-1}\sum_{p=1}^{S}\{\sum_{q=1}^{S} cos[\theta_m(p-q)]\sum_{h=0}^{j-1} u_{Sh+q}\}^2 + \\
& 2J^{-2}\sum_{j=0}^{J-1}\sum_{p=2}^{S}\{\sum_{q=1}^{p-1} cos[\theta_m(p-q)]u_{Sj+q}\}\{\sum_{q=1}^{S} cos[\theta_m(p-q)]\sum_{h=1}^{j} u_{Sh+q}\} + \\
& J^{-2}\sum_{j=0}^{J-1}\sum_{p=2}^{S}\{\sum_{q=1}^{p-1} cos[\theta_m(p-q)]u_{Sj+q}\}^2 \\
&\xrightarrow{L} \int_0^1 [B(r)]' \wedge' A_m' A_m \wedge B(r)dr
\end{aligned}
$$

where

$$
A_m = \begin{pmatrix}
cos(0*\theta_m) & cos(\theta_m) & cos(2\theta_m) & ... & cos((S-1)\theta_m) \\
cos((S-1)\theta_m) & cos(0*\theta_m) & cos(\theta_m) & ... & cos((S-2)\theta_m) \\
cos((S-2)\theta_m) & cos((S-1)\theta_m) & cos(0*\theta_m) & ... & cos((S-3)\theta_m) \\
... & & & & \\
cos(\theta_m*1) & cos(\theta_m*2) & cos(\theta_m*3) & ... & cos(0*\theta_m)
\end{pmatrix}
$$

The convergence of the first term comes from 6.3. The second and the third term converge to 0 considering 6.1 and6.2. Similarly when $m = m_{odd}$, we have $J^{-2}\sum_{t=0}^{T} x_{m,t}^2 \xrightarrow{L} \int_0^1 [B(r)]' \wedge' C_m' C_m \wedge B(r)dr$, where

$$
C_m = \begin{pmatrix}
sin(0) & sin(\theta_m) & sin(2\theta_{m-1}) & ... & sin((S-1)\theta_{m-1}) \\
sin((S-1)\theta_{m-1}) & sin(0*\theta_m) & sin(\theta_{m-1}) & ... & sin((S-2)\theta_{m-1}) \\
sin((S-2)\theta_{m-1}) & sin((S-1)\theta_m) & sin(0*\theta_{m-1}) & ... & sin((S-3)\theta_{m-1}) \\
... & & & & \\
sin(\theta_{m-1}) & sin(2\theta_{m-1}) & sin(3\theta_{m-1}) & ... & sin(0)
\end{pmatrix}.
$$

For the non-diagonal elements in $X'X$, let crg and srg stands for cos or sin and we get:

$$
\begin{aligned}
J^{-2}\sum_{t=0}^{T} x_{m,t}x_{k,t} &= J^{-2}\sum_{j=0}^{J-1}\sum_{p=1}^{S}\{\sum_{q=1}^{S} crg[\theta_m(p-q)]\sum_{h=1}^{j} u_{Sh+q}\}\{\sum_{q=1}^{S} srg[\theta_k(p-q)]\sum_{h=1}^{j} u_{Sh+q}\} + \\
& J^{-2}\sum_{j=0}^{J-1}\sum_{p=2}^{S}\{\sum_{q=1}^{p-1} crg[\theta_m(p-q)]u_{Sj+q}\}\{\sum_{q=1}^{S} srg[\theta_k(p-q)]\sum_{h=1}^{j} u_{Sh+q}\} + \\
& J^{-2}\sum_{j=0}^{J-1}\sum_{p=2}^{S}\{\sum_{q=1}^{p-1} srg[\theta_k(p-q)]u_{Sj+q}\}\{\sum_{q=1}^{S} crg[\theta_m(p-q)]\sum_{h=1}^{j} u_{Sh+q}\} + \\
& J^{-2}\sum_{j=0}^{J-1}\sum_{p=2}^{S}\{\sum_{q=1}^{p-1} crg[\theta_m(p-q)]u_{Sj+q}\}\{\sum_{q=1}^{p-1} srg[\theta_k(p-q)]u_{Sj+q}\}^2 \\
&\rightarrow 0
\end{aligned}
$$

The first term equals to 0 and the rest terms converge to 0 with 6.1 and 6.2.

For the elements in $X'u$, when $m = 1, \pi, m_{even}$:

$$
\begin{aligned}
J^{-1}\sum_{t=0}^{T} x_{m,t}u_t &= J^{-1} \sum_{j=0}^{J-1}\sum_{p=1}^{S}\sum_{q=p-1}^{p-S} cos[\theta_m(p-q)]\sum_{h=0}^{j} u_{Sh+q}u_{Sj+p} \\
&= J^{-1}\sum_{p=1}^{S}\sum_{q=1}^{S} cos[\theta_m(p-q)]\sum_{j=0}^{J-1}\sum_{h=0}^{j} u_{S(h-1)+q}u_{Sj+p} \\
&\quad +J^{-1}\sum_{p=2}^{S}\sum_{q=1}^{p-1} cos[\theta_m(p-q)]\sum_{j=0}^{J-1} u_{Sj+q}u_{Sj+p} \\
&\xrightarrow{L} \int_0^1 [B(r)]' \wedge' A_m \wedge dB(r) + \sum_{p=1}^{S}\sum_{q=1}^{S} cos[\theta_m(p-q)]\Gamma_{p.q} + \\
&\quad \sum_{q=2}^{S}\sum_{p=1}^{q-1} cos[\theta_m(p-q)]\Gamma_{0;p.q}
\end{aligned}
$$

Similarly when $m = m_{odd}$:

$$
\begin{aligned}
J^{-1}\sum_{t=0}^{T} x_{m,t}u_t &\xrightarrow{L} \int_0^1 [B(r)]' \wedge' C_m \wedge dB(r) + \sum_{p=1}^{S}\sum_{q=1}^{S} sin[\theta_{m-1}(p-q)]\Gamma_{p.q} + \\
&\quad \sum_{q=2}^{S}\sum_{p=1}^{q-1} sin[\theta_{m-1}(p-q)]\Gamma_{0;p.q}
\end{aligned}
$$

Denote $J^{-1}\sum_{t=0}^{T} x_{m,t}u_t \xrightarrow{L} h_m$ for all m. Let $\Upsilon = diag(J,...,J)$ to be an $S \times S$ diagonal matrix, $\rho = \Upsilon^{-1}(\Upsilon^{-1}X'X\Upsilon^{-1})^{-1}\Upsilon^{-1}X'u$. We have $\Upsilon^{-1}X'u \xrightarrow{L} [h_1, h_2, ..., h_m]'$ and $\Upsilon^{-1}X'X\Upsilon^{-1} \xrightarrow{L} diag(z_1, z_2, ..., z_s)$, where $z_m = \int_0^1 [B(r)]' \wedge' A_m' A_m \wedge B(r)dr$. The covergence above leads to the asymptotic distribution of the parameter estimates: $J\hat{\rho}_m \xrightarrow{L} \frac{h_m}{z_m}$.

For $\hat{\sigma}_{\rho_m}$, with $s_T^2 \xrightarrow{P} \gamma_0 = \sum_{j=0}^{\infty}\sigma^2\psi_j$ where $\xrightarrow{P}$ denote convergence in probability, we have

$$
\Upsilon^2[\hat{\sigma}^2_{\rho_1}, \hat{\sigma}^2_{\rho_2}, ..., \hat{\sigma}^2_{\rho_S}]' = s_T^2[\Upsilon^{-1}X'X\Upsilon^{-1}]^{-1} \xrightarrow{L} r_0 diag(z_1^{-1}, z_2^{-1}, ..., z_s^{-1})
$$

It leads to $J^2\hat{\sigma}^2_{\rho_m} \xrightarrow{L} r_0 z_m^{-1}$.

With continuous mapping theorem, we get the asymptotic distribution of the t statistics when there are serial correlations in the residuals:

$$
t_m \xrightarrow{L} \frac{h_m}{\{\gamma_0 z_m\}^{1/2}} \tag{6.4}
$$

(II) Simplification:

First, we give the simplification process for h_m and z_m when $m = 1, \pi, m_{even}$. We can see both

$A_m$and $\wedge$ are circulant matrixes, we directly propose the following properties for circulant matrixes:

Properties: Suppose matrix A is a circulant matrix in a form:

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{s-1} \\ a_{s-1} & a_0 & a_1 & \dots & a_{s-2} \\ a_{s-2} & a_{s-1} & a_0 & \dots & a_{s-3} \\ & & & \dots & \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}$$

Then: (a) If both A and C are $S \times S$ circulant matrixes, then A and C commute, e.g., AC=CA.

(b) The eigen values for A is $g_k = \sum_{j=0}^{S-1} a_j e^{\frac{2\pi jk}{S} i}, k = 0, 1, ..., S-1$ and the corresponding eigen vector is $w_k = \frac{1}{\sqrt{S}}[1, e^{\frac{2\pi k}{S} i}, e^{2*\frac{2\pi k}{S} i}, ..., e^{(n-1)*\frac{2\pi k}{S} i}]'$, where i is the imaginary unit.

With (a) we get $\wedge' A_m \wedge = A_m \wedge' \wedge$, and $\wedge' A_m' A_m \wedge = A_m' A_m \wedge' \wedge$. Next with (b) we make eigen value decomposition of $\wedge^* = \wedge' \wedge$. $\wedge^*$ is a symmetric circulant matrix with the first row denoted as $(\wedge_1^*, \wedge_2^*, ..., \wedge_S^*)$, and $\wedge_j^* = \wedge_{S-j}^*$. Its eigen values are $g_k^* = \sum_{j=1}^{S} \wedge_j^* [cos(\frac{2(j-1)\pi k}{S}) + isin(\frac{2(j-1)\pi k}{S})], k = 0, 1, ..., S-1$. Because $\wedge_j^* = \wedge_{S-j}^*$, we can get $\sum_{j=1}^{S} \wedge_j^* sin[\frac{2(j-1)\pi k}{S}] = 0$ for any k. Thus $g_k^* = \sum_{j=1}^{S} \wedge_j^* cos[\frac{2(j-1)\pi k}{S}], k = 0, ..., S-1$.

Because $\sum_{j=1}^{S} \wedge_j^* cos[\frac{2(j-1)\pi k}{S}] = \sum_{j=1}^{S} \wedge_j^* cos[2\pi - \frac{2(j-1)\pi k}{S}]$, w_k and w_{S-k} have the same eigen value except when $k = S-k$. We make the following transformations for the eigen vectors of w_k and w_{S-k}:

$$\varpi_k = \frac{w_k + w_{S-k}}{\sqrt{2}} = \sqrt{\frac{2}{S}}[1, cos\frac{2\pi k}{S}, cos(\frac{4\pi k}{S}), ..., cos(\frac{2(S-1)\pi k}{S})]'$$

$$\varpi_{S-k} = \frac{w_k - w_{S-k}}{\sqrt{2} i} = \sqrt{\frac{2}{S}}[0, sin\frac{2\pi k}{S}, sin(\frac{4\pi k}{S}), ..., sin(\frac{2(S-1)\pi k}{S})]'$$

where $k = 1, 2, ..., \frac{S}{2} - 1$ when S is even, and $k = 1, 2, ..., \frac{S-1}{2}$ when S is odd. For $k = 0$, $\varpi_0 = w_0$ and for $k = \frac{S}{2}$ when S is even, $\varpi_k = w_k$. The eigen value decomposition for $\wedge^*$ is given below:

$$\wedge^* = \widetilde{U} G \widetilde{U}'$$

where $\widetilde{U} = (\varpi_0, \varpi_1, ..., \varpi_{S-1})$, $G = diag(g_0^*, g_1^*, ..., g_{S-1}^*)$.

Next we make singular value decomposition (SVD) for A_m. The details of the singular value decomposition for A_m could be found in Meng and He (2012), we only give a brief statement here. $A_m = U^m D^m (V^m)'$, $D^m = diag(d_0^m, d_1^m, ..., d_{S-1}^m)$ with diagonal elements are square roots of eigen values of $A_m' A_m$. The columns of V^m are the eigen vectors of $A_m' A_m$, denoted as $v_k^m, k = 0, ..., S-1$. because $A_m^T A_m$ is also a symmetric circulant matrix, we could get the columns of V^m are also $\varpi_k, k = 0, 1, ..., (S-1)$. The difference is that the order of ϖ_k is changed because SVD requires that the diagonal elements of D^m are arranged from largest to smallest. The columns of U^m, denoted by $u_k^m, k = 0, ..., S-1$, are derived by $u_k^m = (d_k^m)^{-1} A_m v_k^m$.

We re-arrange the columns of $\widetilde{U}$ in $\wedge^*$ to make the columns the same as V^m, the diagonal elements of Υ are also arranged accordingly, denote the new diagonal matrix as $\widetilde{G}$ with diagonal elements $\widetilde{g}_k, k = 0, 1, ..., S-1$, so $\wedge^* = V^m \widetilde{G} (V^m)'$. Considering $(V^m)' V^m$ is identity matrix, we get the following simplification:

$$A_m \wedge^* = U^m D^m \widetilde{G}(V^m)' \quad A_m' A_m \wedge^* = U^m D^m D^m \widetilde{G}(V^m)'$$

For h_m, when $m = 1$, $d_0^1 = S$, $d_k^1 = 0, k = 1, 2, ..., S-1$, only the first column of V^1 matters, $u_0^1 = v_0^1 = \frac{1}{\sqrt{s}}(1, 1, ..., 1)'$. The eigen value of $\wedge^*$ corresponding to v_0^1 is $\sum_{j=1}^S \wedge_j^*$. We have $d_0^1 \widetilde{g}_0 = S \sum_{j=1}^S \wedge_j^*$, $d_k^1 \widetilde{g}_k = 0, k = 1, 2, ..., S-1$. For $B(r) = [B_1(r), B_2(r), ..., B_S(r)]'$ where $B_j(r), j = 1, .., S$ are mutually independent standard Brownian motions, we could write $B'(r) u_0^1 = (v_0^1)' B(r) = W_1(r)$, $W_1(r)$ is a standard Brownian motion. The decompositions above lead to the equation:

$$\int_0^1 [B(r)]' \wedge' A_1 \wedge dB(r) = \int_0^1 [B(r)]' A_1 \wedge^* dB(r) = \int_0^1 [B(r)]' U^1 D^1 \widetilde{G}(V^1)' dB(r)$$

$$= \sum_{j=1}^S S \wedge_j^* \int_0^1 W_1(r) dW_1(r)$$

Similarly for z_m:

$$\int_0^1 [B(r)]' \wedge' A_1' A_1 \wedge B(r) dr = \sum_{j=1}^S S^2 \wedge_j^* \int_0^1 W_1(r)^2 dr$$

When $m = \pi$, $d_0^\pi = S$, $d_k^\pi = 0, k = 1, 2, ..., S-1$, only $v_0^\pi = \frac{1}{\sqrt{s}}(1, -1, ..., 1, -1)'$ matters. The eigen value of $\wedge^*$ corresponding to v_0^π is $\sum_{j=1}^S \wedge_j^* cos[(j-1)\pi] = \sum_{j=1}^S (-1)^{j-1} \wedge_j^*$, so $d_0^\pi \widetilde{g}_0 = S \sum_{j=1}^S (-1)^{j-1} \wedge_j^*$, $d_k^1 \widetilde{g}_k = 0, k = 1, 2, ..., S-1$. We have $B'(r) u_0^m = (v_0^m)' B(r) = W_m(r)$ and it lead to the equation:

$$\int_0^1 [B(r)]' \wedge' A_m \wedge dB(r) = \sum_{j=1}^S (-1)^{j-1} S \wedge_j^* \int_0^1 W_m(r) dW_m(r)$$

Similarly for z_m:

$$\int_0^1 [B(r)]' \wedge' A_m' A_m \wedge B(r) dr = \sum_{j=1}^S (-1)^{j-1} S^2 \wedge_j^* \int_0^1 W_2(r)^2 dr$$

When $m = m_{even}$, $d_0^m = d_1^m = \frac{S}{2}$, $d_k^m = 0, k = 2, 3, ..., S-1$, $u_0^m = v_0^m = \sqrt{\frac{2}{S}}[1, cos\theta_m, cos(2\theta_m), ..., cos((S-1)\theta_m)]'$, $u_1^m = v_1^m = \sqrt{\frac{2}{S}}[0, sin\theta_m, sin(2\theta_m), ..., sin((S-1)\theta_m)]'$, the corresponding eigen value of $\wedge^*$ is $\sum_{j=1}^S \wedge_j^* cos[(j-1)\theta_m]$. So $d_0^m \widetilde{g}_0 = d_1^m \widetilde{g}_1 = S \sum_{j=1}^S \wedge_j^* cos[(j-1)\theta_m]$, $d_k^1 \widetilde{g}_k = 0, k = 2, 3, ..., S-1$. With the results above we have $B'(r) u_0^m = (v_0^m)' B(r) = W_m(r)$, $B'(r) u_1^m = (v_1^m)' B(r) = W_{m+1}(r)$, and it lead to the equation:

$$\int_0^1 [B(r)]' \wedge' A_m \wedge dB(r) = \sum_{j=1}^S \frac{S}{2} \wedge_j^* cos[(j-1)\theta_m] [\int_0^1 W_m(r) dW_m(r) + \int_0^1 W_{m+1}(r) dW_{m+1}(r)]$$

For z_m:

$$\int_0^1 [B(r)]' \wedge' A_m' A_m \wedge B(r)dr = \sum_{j=1}^{S} \frac{S^2}{4} \wedge_j^* cos[(j-1)\theta_m][\int_0^1 W_m(r)^2 dr + \int_0^1 W_{m+1}(r)^2 dr]$$

Substitute the terms above to (6.4), the distributions of t statistics when $m = 1, \pi, m_{even}$ are derived.

When $m = m_{odd}$, the procedure is the same but with A_m changed to C_m. For SVD of C_m: $C_m = U^m D^m V^m$, we have $d_0^m = d_1^m = \frac{S}{2}$, $d_k^m = 0, k = 2, 3, ..., S-1$, the corresponding eigen value of $\wedge^*$ is $\sum_{j=1}^{S} \wedge_j^* cos[(j-1)\theta_{m-1}]$. Only the first 2 columns of U^m and V^m matter. We have $v_0^m = v_0^{m-1} = \sqrt{\frac{2}{S}}[1, cos\theta_m, cos(2\theta_m), ..., cos((S-1)\theta_m)]'$, $v_1^m = v_1^{m-1} = \sqrt{\frac{2}{S}}[0, sin\theta_m, sin(2\theta_m), ..., sin((S-1)\theta_m)]'$. With the relationship $u_j^m = \lambda_j^{-1} C_m v_j^m$, it is obtained $u_0^m = -v_1^m$, $u_1^m = v_0^m$. Thus we get the simlification needed for h_m:

$$\int_0^1 [B(r)]' \wedge' C_m \wedge dB(r) = \sum_{j=1}^{S} \frac{S}{2} \wedge_j^* cos[(j-1)\theta_{m-1}][\int_0^1 W_m(r)dW_{m-1}(r) - \int_0^1 W_{m-1}(r)dW_m(r)]$$

For z_m,

$$\int_0^1 [B(r)]' \wedge' C_m' C_m \wedge B(r)dr = \sum_{j=1}^{S} \frac{S^2}{4} \wedge_j^* cos[(j-1)\theta_{m-1}][\int_0^1 W_{m-1}(r)^2 dr + \int_0^1 W_m(r)^2 dr]$$

The distributions of t statistics when $m = m_{odd}$ are derived.

Appendix 1.2

To proof Theorem 2: The following Lemma is needed.

Lemma A1:

$$(a)\ \frac{1}{\sigma^2 S} \sum_{p=1}^{S} \bar{x}_{m,p} \bar{\varepsilon}_p \xrightarrow{L} \begin{cases} \sum_{j=1}^{S} \wedge_j^* W_1(1) \int_0^1 W_1(r)dr & m = 1 \\ \sum_{j=1}^{S} \wedge_j^* cos[(j-1)\pi] \int_0^1 W_2(r)dr & m = \pi \\ \frac{1}{2} \sum_{j=1}^{S} \wedge_j^* cos[(j-1)\theta_m][W_m(1) \int_0^1 W_m(r)dr + & \\ W_{m+1}(1) \int_0^1 W_{m+1}(r)dr] & m = m_{even} \\ \frac{1}{2} \sum_{j=1}^{S} \wedge_j^* cos[(j-1)\theta_{m-1}]\{W_m(1) \int_0^1 W_{m-1}(r)dr - & \\ W_{m-1}(1) \int_0^1 W_m(r)dr\} & m = m_{odd} \end{cases}$$

$(b)\ \frac{1}{\sigma^2 TS}\sum_{p=1}^{S}\bar{x}_{m,p}^2 \xrightarrow{L} \begin{cases} \sum_{j=1}^{S}\wedge_j^*(\int_0^1 W_1(r)dr)^2 & m=1 \\ \sum_{j=1}^{S}\wedge_j^* cos[(j-1)\pi][\int_0^1 W_2(r)dr]^2 & m=\pi \\ \frac{1}{4}\sum_{j=1}^{S}\wedge_j^* cos[(j-1)\theta_m]\{[\int_0^1 W_m(r)dr]^2+[\int_0^1 W_{m+1}(r)dr]^2\} & m=m_{even} \\ \frac{1}{4}\sum_{j=1}^{S}\wedge_j^* cos[(j-1)\theta_{m-1}]\{[\int_0^1 W_{m-1}(r)dr]^2+[\int_0^1 W_m(r)dr]^2\} & m=m_{odd} \end{cases}$

$(c)\ J^{-1}\sum_{p=1}^{S}\overline{x}_{m,p}\overline{x}_{k,p} \xrightarrow{L} 0$

where $\overline{\varepsilon}_p = J^{-1}\sum_{j=0}^{J-1}\varepsilon_{Sj+p}$, $\overline{x}_{m,p} = J^{-1}\sum_{j=0}^{J-1}x_{m,Sj+p}$,

Proof (a): The following convergence is used:

$$J^{-1/2}\sum_{j=1}^{J}U_j \xrightarrow{L} \wedge B(1) \tag{6.5}$$

$$J^{-3/2}\sum_{j=1}^{J}\Xi_{j-1} \xrightarrow{L} \wedge\int_0^1 B(r)dr \tag{6.6}$$

We have the left part of (a):

$$\begin{aligned} T^{-1}J\sum_{p=1}^{S}\overline{x}_{m,p}\overline{u}_p &= T^{-1}J\sum_{p=1}^{S}(J^{-1}\sum_{j=0}^{J-1}x_{m,Sj+p})(J^{-1}\sum_{j=0}^{J-1}u_{Sj+p}) \\ &= \frac{1}{S}\sum_{p=1}^{S}(J^{-3/2}\sum_{j=0}^{J-1}x_{m,Sj+p})(J^{-1/2}\sum_{j=0}^{J-1}u_{Sj+p}) \end{aligned}$$

When $m=1,\pi,m_{even}$, consider $J^{-3/2}\sum_{j=0}^{J-1}x_{m,Sj+p}$:

$$\begin{aligned} \sum_{j=0}^{J-1}x_{m,Sj+p} &= \sum_{j=0}^{J-1}\sum_{q=p-1}^{p-S}cos[\theta_m(p-q)]\sum_{h=0}^{j}u_{Sh+q} \\ &= \sum_{q=1}^{S}cos[\theta_m(p-q)]\sum_{j=0}^{J-1}\sum_{h=0}^{j}u_{S(h-1)+q} + \sum_{q=1}^{p-1}cos[\theta_m(p-q)]\sum_{j=0}^{J-1}u_{Sj+q} \end{aligned}$$

According to 6.5, $J^{-3/2}\sum_{q=1}^{p-1}cos[\theta_m(p-q)]\sum_{j=0}^{J-1}u_{Sj+q} \xrightarrow{P} 0$, thus we get the left part of (a) in the following form:

$$S^{-1}\sum_{p=1}^{S}\overline{x}_{m,p}\overline{u}_p = \frac{1}{S}\sum_{p=1}^{S}\sum_{q=1}^{S}\{cos[\theta_m(p-q)]J^{-\frac{3}{2}}\sum_{j=0}^{J-1}\sum_{h=0}^{j}u_{S(h-1)+q}\}(J^{-\frac{1}{2}}\sum_{j=0}^{J-1}u_{Sj+p})$$

Also from 6.5 we get $J^{-\frac{1}{2}}\sum_{j=0}^{J-1} u_{Sj+p} \xrightarrow{L} \sum_{i=1}^{S} \wedge_{p,i} B_i(1)dr$. According to 6.6 we get $J^{-3/2}\sum_{j=0}^{J-1}\sum_{h=0}^{j} u_{S(h-1)+q} \xrightarrow{L} \sum_{i=1}^{S} \wedge_{q,i}\int_0^1 B_i(r)dr$. Thus with the 2 convergences we get the following convergence:

$$\sum_{p=1}^{S} \overline{x}_{m,p}\overline{u}_p \xrightarrow{L} [\int_0^1 B(r)dr]' \wedge' A_m \wedge B(r)$$

When $m = m_{odd}$, the process is the same with above, and we get:

$$\sum_{p=1}^{S} \overline{x}_{m,p}\overline{u}_p \xrightarrow{L} [\int_0^1 B(r)dr]' \wedge' C_m \wedge B(r)$$

With the same simplification method in the Proof of Theorem 1, (a) is proofed.

Proof (b): In (a), replace $\overline{u}_p$ with $\overline{x}_{m,p}$, and we could get for $m = 1, \pi, m_{even}$,

$$J^{-1}\sum_{p=1}^{S} \overline{x}^2_{m,p} \xrightarrow{L} [\int_0^1 B(r)dr]' \wedge' A'_m A_m \wedge B(r)$$

For $m = m_{odd}$: we get $J^{-1}\sum_{p=1}^{S} \overline{x}^2_{m,p} \xrightarrow{L} [\int_0^1 B(r)dr]' \wedge' C'_m C_m \wedge B(r)$

Again, use the same decomposition method in the Proof of Theorem 1, (b) is proofed.

Proof (c): Again, crg and srg stands for cos or sin depending on m and k.

$$\begin{aligned}
J^{-1}\sum_{p=1}^{S} \overline{x}_{m,p}\overline{x}_{k,p} &= J^{-1}\sum_{p=1}^{S}(J^{-1}\sum_{j=0}^{J-1} x_{m,Sj+p})(J^{-1}\sum_{j=0}^{J-1} x_{k,Sj+p}) \\
&= \sum_{p=1}^{S}\{\sum_{q=1}^{S} crg[\theta_m(p-q)]J^{-\frac{3}{2}}\sum_{j=0}^{J-1}\sum_{h=1}^{j} u_{S(h-1)+q}\}\{\sum_{q=1}^{S} srg[\theta_k(p-q)]J^{-\frac{3}{2}}\sum_{j=0}^{J-1}\sum_{h=1}^{j} u_{S(h-1)+q}\} + \\
&\quad J^{-1}\sum_{p=2}^{S}\{\sum_{q=1}^{p-1} crg[\theta_m(p-q)]J^{-\frac{1}{2}}\sum_{j=0}^{J-1} u_{Sj+q}\}\{\sum_{q=1}^{S} srg[\theta_k(p-q)]J^{-\frac{3}{2}}\sum_{j=0}^{J-1}\sum_{h=1}^{j} u_{S(h-1)+q}\} + \\
&\quad J^{-1}\sum_{p=2}^{S}\{\sum_{q=1}^{p-1} srg[\theta_k(p-q)]J^{-\frac{1}{2}}\sum_{j=0}^{J-1} u_{Sj+q}\}\{\sum_{q=1}^{S} crg[\theta_m(p-q)]J^{-\frac{3}{2}}\sum_{j=0}^{J-1}\sum_{h=1}^{j} u_{S(h-1)+q}\} + \\
&\quad J^{-2}\sum_{p=2}^{S}\{\sum_{q=1}^{p-1} crg[\theta_m(p-q)]J^{-\frac{1}{2}}\sum_{j=0}^{J-1} u_{Sj+q}\}\{\sum_{q=1}^{p-1} srg[\theta_k(p-q)]J^{-\frac{1}{2}}\sum_{j=0}^{J-1} u_{Sj+q}\}
\end{aligned}$$

The first term equals to 0 and the rest terms converge to 0 according to 6.5 and 6.6. (c) is proofed.

Next we give proof of Theorem 2:

Regress y_t, $x_{m,t}$ and u_t on the deterministic component included in (3.4), we could re express (3.4) as

$$(1-B^S)\widetilde{y}_t = \sum_{m=1}^{S} \rho_m \widetilde{x}_{m,t} + \widetilde{u}_t \tag{6.7}$$

where $\widetilde{y}_t = y_t - c_{1,y} - \sum_{i=2}^{S} c_{i,y} D_{i,t}$, $\widetilde{x}_{m,t} = x_{m,t} - c_{1,m} - \sum_{i=2}^{S} c_{i,m} D_{i,t}$, $\widetilde{u}_t = u_t - c_{1,\varepsilon} - \sum_{i=2}^{S} c_{i,\varepsilon} D_{i,t}$.
We could express the terms above with trend and seasonal means:
$\widetilde{x}_{m,t} = x_{m,t} - \overline{x}_{m,p}$, $\overline{x}_{m,p} = J^{-1}\sum_{j=1}^{J} x_{m,Sj+p}$ with $t = Sj + p$
$\widetilde{u}_t = u_t - \overline{u}_p$, $\overline{u}_p = J^{-1}\sum_{j=1}^{J} u_{Sj+p}$.

For 6.7, the t statistics could be expressed as $t_m = \frac{\hat{\rho}_m}{\hat{\sigma}_{\rho_m}}$. Similar with the proof of Theorem, the OLS estimator $\Upsilon^{-1}\hat{\rho} = [\Upsilon^{-1}\widetilde{X}'\widetilde{X}\Upsilon^{-1}]^{-1}\Upsilon^{-1}\widetilde{X}\varepsilon$, where $\widetilde{X} = [\widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_S]$ with $\widetilde{x}_m = [\widetilde{x}_{m,1}, \widetilde{x}_{m,2}, ..., \widetilde{x}_{m,T}]'$.

For the elements in $\Upsilon^{-1}\widetilde{X}\varepsilon$:

$$\begin{aligned} J^{-1}\sum_{t=1}^{T}\widetilde{x}_{m,t}\widetilde{u}_t &= J^{-1}\sum_{t=0}^{T}(x_{m,t}-\overline{x}_{m,p})(u_t-\overline{u}_p) \\ &= J^{-1}(\sum_{t=1}^{T}x_{m,t}u_t - \sum_{t=1}^{T}x_m\overline{u}_p - \sum_{t=1}^{T}\overline{x}_{m,p}u_t + \sum_{t=1}^{T}\overline{x}_{m,p}\overline{u}_p) \\ &= J^{-1}\sum_{t=1}^{T}x_{m,t}u_t - \sum_{p=1}^{S}\overline{x}_{m,p}\overline{u}_p \end{aligned}$$

The last step comes from $\sum_{t=1}^{T} x_{m,t}\overline{u}_p = \sum_{t=1}^{T}\overline{x}_{m,p}u_t = \sum_{t=1}^{T}\overline{x}_{m,p}\overline{u}_p = J\sum_{p=1}^{S}\overline{x}_{m,p}\overline{u}_p$. The convergences of $J^{-1}\sum_{t=1}^{T} x_{m,t}u_t$ are given in the proof of Theorem 1, and the convergences of $\sum_{p=1}^{S}\overline{x}_{m,p}\overline{u}_p$ are given in Lemma A1 (a). Then the convergence of $J^{-1}\sum_{t=1}^{T}\widetilde{x}_{m,t}\widetilde{u}_t$ is derived, denote $J^{-1}\sum_{t=1}^{T}\widetilde{x}_{m,t}\widetilde{u}_t \xrightarrow{L} \widetilde{h}_m$, m=1,2,...,S.

Next consider the elements in $\Upsilon^{-1}\widetilde{X}'\widetilde{X}\Upsilon^{-1}$. For non-diagonal elements:

$$\begin{aligned} J^{-2}\sum_{t=1}^{T}\widetilde{x}_{m,t}\widetilde{x}_{k,t} &= J^{-2}\sum_{t=0}^{T}(x_{m,t}-\overline{x}_{m,p})(x_{k,t}-\overline{x}_{k,p}) \\ &= J^{-2}(\sum_{t=1}^{T}x_{m,t}x_{k,t} - \sum_{t=1}^{T}x_{m,t}\overline{x}_{k,p} - \sum_{t=1}^{T}\overline{x}_{m,p}x_{k,t} + \sum_{t=1}^{T}\overline{x}_{m,p}\overline{x}_{k,p}) \\ &= J^{-2}\sum_{t=1}^{T}x_{m,t}x_{k,t} - J^{-1}\sum_{p=1}^{S}\overline{x}_{m,p}\overline{x}_{k,p} \end{aligned}$$

The last step comes from $\sum_{t=1}^{T} x_{m,t}\overline{x}_{k,p} = \sum_{t=1}^{T}\overline{x}_{m,p}x_{k,t} = \sum_{t=1}^{T}\overline{x}_{m,p}\overline{x}_{k,p} = J\sum_{p=1}^{S}\overline{x}_{m,p}\overline{x}_{k,p}$. The first term are shown in proof of Theorem 1 that it converges to 0, and the second term converges to 0 according to Lemma A1 (c). Thus $J^{-2}\sum_{t=1}^{T}\widetilde{x}_{m,t}\widetilde{x}_{k,t}\xrightarrow{L}0$.

For diagonal elements:

$$\begin{aligned}J^{-2}\sum_{t=1}^{T}\widetilde{x}_{m,t}^2 &= J^{-2}\sum_{t=0}^{T}(x_{m,t}-\overline{x}_{m,p})^2\\ &= J^{-2}(\sum_{t=1}^{T}x_{m,t}x_{k,t}-2\sum_{t=1}^{T}x_{m,t}\overline{x}_{m,p}+\sum_{t=1}^{T}\overline{x}_{m,p}^2)\\ &= J^{-2}\sum_{t=1}^{T}x_{m,t}^2-J^{-1}\sum_{p=1}^{S}\overline{x}_{m,p}^2\end{aligned}$$

The convergence of the 2 terms are given in proof of Theorem 1 and Lemma A1 (b) and thus could be derived directly. Denote $J^{-2}\sum_{t=1}^{T}\widetilde{x}_{m,t}^2\xrightarrow{L}\widetilde{z}_m$, m=1,2,...,S.

With the proof above, we have $J\hat{\rho}_m=\frac{\widetilde{h}_m}{\widetilde{z}_m}$. Similarly, we get the convergence of the standard error of the estimators: $J\hat{\sigma}_{\rho_m}\xrightarrow{L}\sqrt{\frac{\gamma_0}{\widetilde{z}_m}}$. Thus we get the asymptotic distributions of the t statistics: $t_m=\frac{J\hat{\rho}_m}{J\hat{\sigma}_{\rho_m}}\xrightarrow{L}\frac{\widetilde{h}_m}{\sqrt{\gamma_0\widetilde{z}_m}}$.

A1.3 Asymptotic distribution of corrected t statistics.

We do not give the proof for the case when $m=m_{odd}$, the procedure is the same with $m=m_{even}$. For $m=1,\pi,m_{even}$, we give the following proof.

With this term above the asymptotic distribution of t statistics could be expressed as follows:

$$\begin{aligned}t_m &- \frac{\sum_{p=1}^{S}\sum_{q=1}^{S}cos(\theta_m(p-q))e_p'\Gamma e_q+\sum_{q=2}^{S}\sum_{p=1}^{q-1}cos(\theta_m(p-q))e_p'\Gamma_0 e_q}{\{\gamma_0\int_0^1[B(r)]'\wedge' A_m'A_m\wedge B(r)dr\}^{1/2}}\\ &\xrightarrow{L} \frac{\int_0^1[B(r)]'\wedge' A_m\wedge dB(r)}{\{\gamma_0\int_0^1[B(r)]'\wedge' A_m'A_m\wedge B(r)dr\}^{1/2}}\end{aligned}$$

We have the convergence $s_T^2\xrightarrow{P}\gamma_0$, where $\xrightarrow{P}$ denote convergence in probability. More over, consider the asymptotic distributions of the variance of parameter estimators:

$$J^2\hat{\sigma}_{\rho_m}^2=\frac{s_T^2}{J^{-2}\sum_{t=0}^{T}x_{m,t}^2}\xrightarrow{L}\frac{\gamma_0}{\int_0^1[B(r)]^T\wedge' A_m'A_m\wedge B(r)dr}$$

Then we have the following convergence:

$$t_m - (\frac{J\hat{\sigma}_{\rho_m}}{s_T})\frac{\sum_{p=1}^{S}\sum_{q=1}^{S} cos(\theta_m(p-q))e_p^{'}\Gamma e_q + \sum_{q=2}^{S}\sum_{p=1}^{q-1} cos(\theta_m(p-q))e_p^{'}\Gamma_0 e_q}{\gamma_0^{1/2}}$$

$$\xrightarrow{L} \frac{\int_0^1 [B(r)]' \wedge' A_m \wedge dB(r)}{\{\gamma_0 \int_0^1 [B(r)]' \wedge' A_m' A_m \wedge B(r)dr\}^{1/2}} \tag{6.8}$$

With the simplification process in Appendix 1.1, multiply both sides by $(\frac{\gamma_0}{\sum_{j=1}^{S} e_j' \wedge^* e_j cos((j-1)\theta_m)})^{\frac{1}{2}}$ and substitute γ_0, Γ, Γ_0 with their estimates s_T^2, $\hat{\Gamma}$, $\hat{\Gamma}_0$, the proof is finished.

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