DALARNA
UNIVERSITY

Working papers in transport, tourism, information technology and microdata analysis

A stopping rule while searching for optimal solution of facility-location

Författare 1: Kenneth Carling
Författare 2: Xiangli Meng
Editor: Hasan Fleyeh

Nr: 2013:20

Working papers in transport, tourism, information technology and microdata analysis
ISSN: 1650-5581

A stopping rule while searching for optimal solution of facility-location

Authors♦: Kenneth Carling and Xiangli Meng♣

Abstract: Solutions to combinatorial optimization, such as *p*-median problems of locating facilities, frequently rely on heuristics to minimize the objective function. The minimum is sought iteratively and a criterion is needed to decide when the procedure (almost) attains it. However, pre-setting the number of iterations dominates in OR applications, which implies that the quality of the solution cannot be ascertained. A small branch of the literature suggests using statistical principles to estimate the minimum and use the estimate for either stopping or evaluating the quality of the solution. In this paper we use test-problems taken from Baesley's OR-library and apply Simulated Annealing on these *p*-median problems. We do this for the purpose of comparing suggested methods of minimum estimation and, eventually, provide a recommendation for practioners. An illustration ends the paper being a problem of locating some 70 distribution centers of the Swedish Post in a region.

Key words: *p*-median problem, Simulated Annealing, discrete optimization, extreme value theory

---

♦ Kenneth Carling is a professor in Statistics and Xiangli Meng is a PhD-student in Micro-data analysis at the School of Technology and Business Studies, Dalarna university, SE-791 88 Falun, Sweden. We are grateful to partcipants at INFORMS Euro 2013 in Rome for useful comments on a previous version. Financial support from the Swedish Retail and Wholesale Development Council is gratefully acknowledged.

♣ Corresponding author. E-mail: xme@du.se. Phone: +46-23-778509.

1. Introduction

Consider the problem of finding a solution to $\min_{\Theta} f(\Theta)$ where the complexity of the function renders analytical solutions infeasible. To focus ideas, consider the *p*-median problem on the network. The problem is to allocate P facilities to a population geographically distributed in Q demand points such that the population's average or total distance to its nearest service center is minimized. Hakimi (1964) considered the task of locating telephone switching centers and showed later (Hakimi, 1965) that, in a network, the optimal solution of the *p*-median model existed at the nodes of the network. If *N* is the number of nodes, then the number of possible locations of facilities amounts to $\binom{N}{P}$ which often is an astronomically large number rendering an enumeration of all possibilities infeasible.

To overcome that enumeration is infeasible, much research has been devoted to efficient (heuristic) algorithms to solve the *p*-median model (see Handler and Mirchandani, 1979 and Daskin, 1995 as examples). In this work we will rely on a common heuristic known as Simulated Annealing which is well described by Lenanova and Loresh (2004). The virtue of Simulated Annealing as other heuristics is that the algorithm will iterate towards a good solution, not necessarily the actual optimum.

The overall aim of this paper is to provide a practical rule for determining when a good solution is found and the algorithm may be stopped. While there is an ample literature on heuristic algorithms, only a few papers address the stopping criterion. Accordingly, the prevailing practice is to run the heuristic algorithm for a pre-specified number of iterations or until improvements in the solution becomes infrequent. Such practice does not lend itself to determine the quality of the solution in a specific problem and is therefore unsatisfactory.

This paper is organized as follows: in section two we review suggested methods for statically estimating the minimum of the objective function and add some further remarks on the issue. In the third section we compare the methods by applying them to test-problems of varying complexity, taken from the Beasley OR-library (Beasley, 1990). In section four we provide a recommendation for a stopping rule being, at least, applicable on *p*-median problems tackled by Simulated Annealing.

Section five presents a real-world problem of locating distribution centres for the Swedish Post in a region in mid-Sweden. The sixth section concludes this paper.

2. Statistical estimation of the minimum of an objective function

When solving a *p*-median problem, a heuristic algorithm is supposed to stop when the objective function is close to its minimum. In practice, the minimum is unknown. A possible solution might be to statistically estimate the minimum, and a few approaches have been proposed for this purpose. Before explaining them, we introduce the notations used throughout the paper:

z_p = feasible solution of locating *P* facilities in *N* nods, indexed by p, $p = 1,2, \dots, \binom{N}{P}$.

A = the set of all feasible solutions $A = \left\{z_1, z_2, \dots, z_{\binom{N}{P}}\right\}$.

$g(z_p)$ = the objective function of solution z_p.

$\theta = \min_A g(z_p)$.

$\hat{\theta}$ = an estimator of θ.

$g_{(i)}$ = the i^{th} smallest value in a random subsample A_n of size n, $i = 1,2, \dots, n$.

$\tilde{x}_r^s$ = the heuristic solution in the r^{th} iteration of the s^{th} sample.

There are mainly two approaches that are proposed for estimating the minimum being the Jackknifing approach (hereafter JK) and extreme value theory approach (EVT). The JK-estimator is introduced by Quenouille (1956):

$$\hat{\theta}_{JK} = \sum_{i=0}^{J} (-1)^i \binom{J+1}{i+1} g_{(i+1)}$$

where J is the order. Dannenbring (1977) and Nydick and Weiss (1988) suggest to use the first order, i.e. $J = 1$, for point estimating the minimum. Their argument is a bias of order n^{-2} and a mean square error being lower for the first order compared with higher orders as shown by Robson and Whitlock (1964).

Derigs (1985) discusses using EVT to estimate the optimum. In contrast to the JK-approach of choosing one random sample, he chooses S random samples and considers $g_{(1)}$ of each sample an extreme value assumed to follow the Weibull distribution. The EVT-estimator of the minimum is the smallest value of the s extreme values, and Derigs (1985) also derives a confidence interval for the minimum. Later Wilson, King and Wilson (2004), referring to others work, employ the idea of substituting the extreme values obtained from a random sample by those produced as best solutions in s runs of a heuristics, i.e. $\tilde{x}_r^s$. In that case a $100(1-e^{-s})\%$ confidence interval is found as $\left[\min(\tilde{x}_r^s) - \hat{b}, \min(\tilde{x}_r^s)\right]$, where $\hat{b}$ is the estimated shape parameter of the Weibull distribution. Consequently, the EVT approach offers a measure of uncertainty of its estimator in contrast to the JK approach.

Table 1 gives an example of the two approaches by means of the first p-median problem in the OR-library presented in Beasley (1990). For the illustration we randomly picked z_p of size 100, computed the corresponding objective functions $g(z_p)$, and obtained $\hat{\theta}_{JK} = 2g_{(1)} - g_{(2)}$. To illustrate the EVT approach, we set $S = 10$ with each one based on a random sample of size $n = 100$ from A. The smallest value in each of $S = 10$ cases is used to estimate the parameters of the Weibull distribution. In line with Wilson et al. (2004), we used least squares estimation of the Weibull parameters with the Nelder-Mead simplex search procedure. For $s = 10$, the level of the confidence interval is presumed to be $100(1-e^{-s})\% \approx 99.99\%$.

Table 1: Illustration of the JK and EVT approaches on the 1st OR-lib problem.

	Optimum	Estimator	St. Dev.	99.99%-CI
JK	5817	6650	*68.26*	*[6445—6707]*
EVT	5817	6339	*157.52*	[-1256—6339]

Note: In italics are complementary quantities suggested by us.

Neither of the approaches offers an estimate of the standard deviation of the estimator. We suggest the following methods to estimate their standard deviations and the 99% confidence interval for JK estimator. The bootstrap method by Efron (1979) is proposed to estimate it for the JK-estimator. Furthermore, we propose as a confidence interval $[\hat{\theta}_{JK} - 3\sigma^*(\hat{\theta}_{JK}), \min g(z_p)]$, where $\sigma^*(\hat{\theta}_{JK})$ is the standard deviation of $\hat{\theta}_{JK}$ obtained from bootstrapping. With the scalar 3, the level corresponds to 99.9% provided that the sampling distribution of the estimator being Normal. The quantities drawing on bootstrapping are shown in Table 1 as well as the standard deviation of the EVT-estimator obtained from the *s* sample-minima, and they are given in italics.

The JK approach is known to perform poorly, as also evident in this example with an estimator 10% off the actual minimum (see Nydick and Weiss, 1988). The problem lays in the required size of the random sample. The objective function in p-median problems might be regarded as approximately Normal with a truncation in the left tail being the minimum θ. A good estimate of θ would require a random sample with some values near to θ. For θ far out in the tail, the required sample size to get such values would be huge. We show below that for many of the OR-library p-median problems, the minimum is at least some 6 standard deviations away from the mean requiring a sample size of $1/\Phi(-6) \approx 10^9$ (Φ is the standard Normal distribution function) to render hope of obtaining a random sample containing values close to θ. Such a computational effort is better spent at searching for the minimum by means of an effective heuristic.

Fortunately, several authors point out that, if the starting values are picked at random, repeated heuristic solutions mimic a random sample in the tail (see e.g. McRoberts, 1971 and Golden and Alt, 1979). Thereby random values in the tail can be obtained at a much less computational effort and used for estimating θ. This goes for both the JK-estimator as for the EVT-estimator.

The JK-estimator is nonparametric. The rationale for the EVT approach and that the heuristic solutions follow the Weibull distribution comes from a belief that $g\left(z_p\right)$ follows a skewed (or a uniform distribution). However, we took large random subsamples of *A* for the 40 problems in the OR-library and found that $g\left(z_p\right)$ is typically symmetrically distributed and only slightly skewed in instances *P* small. Figure 1 shows an example being the 14$^{\text{th}}$ problem in OR-library: one million z_p:s are drawn at random from A and the histogram of the corresponding $g(z_p)$ is given. This empirical distribution and those of the other problems suggest it more reasonable to expect $g(z_p)$ to follow a Normal distribution. As a consequence, the extreme values would better be modelled by the Gumbel distribution (see Kotz and Nadarajah, 2000 (p. 59)).

The choice of Gumbel or Weibull distribution will not affect the EVT-estimator and its variance, but the confidence interval. In the Gumbel case, the confidence interval is derived by the theoretic percentile $[\mu - \sigma \ln(-\ln(1-\alpha)), \min(\tilde{x}_r^s)]$ where μ and σ are the location and shape parameters of the Gumbel distribution while α is the confidence level. The parameters may be estimated by the moments as $\hat{\sigma} = \frac{\sqrt{6\text{var}(\tilde{x}_r^s)}}{\pi}$ and $\hat{\mu} = mean(\tilde{x}_r^s) - 0.57722\hat{\sigma}$ where details are in Kotz and Nadarajah (2000, p. 12).

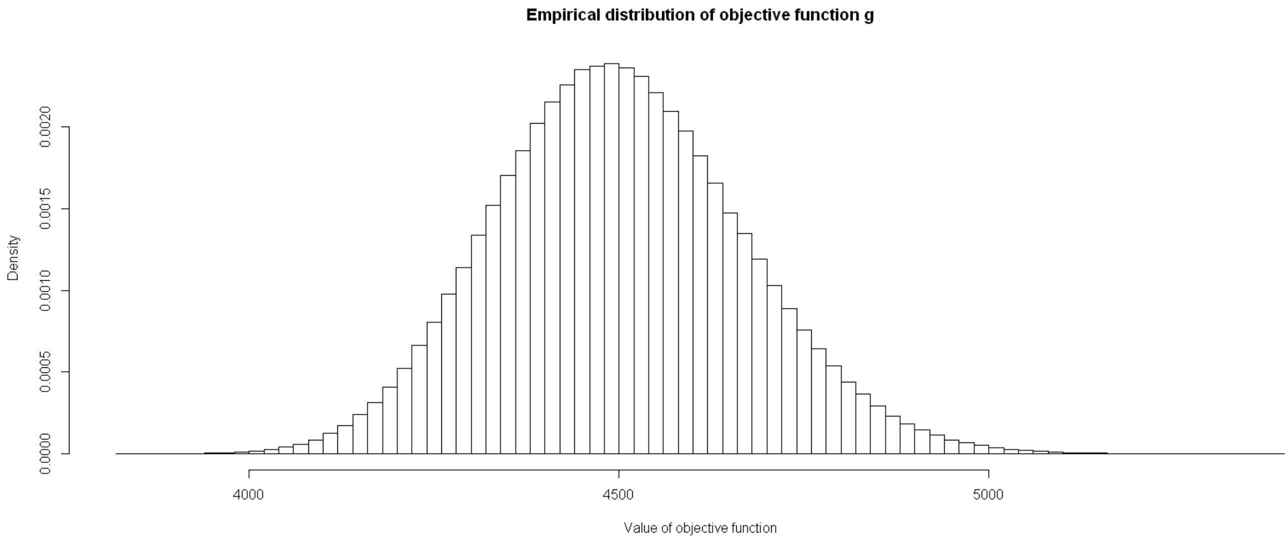


Figure 1: Sample distribution of the 14th problem in the OR-library

3. Experimental comparison of the three estimators

The two EVT-estimators with *W* for Weibull and *G* for Gumbel $\hat{\theta}_W$, $\hat{\theta}_G$, and the JK-estimator $\hat{\theta}_{JK}$ are justified by different arguments and there is no way to deem one superior to the other unless they are put to test. We have not found any previous work comparing the estimators so we use the OR-library problems for an experimental comparison of them. We present the results of a selection of 6 of the problems here, deferring the results of the remaining 34 to the Appendix. The minimum θ of the 40 problems are known and the problems vary substantially in complexity as we define here as $\left(\left(\mu_{g(z_p)} - \theta\right) / \sigma_{g(z_p)} \right)$. In Table 2 we give θ, $\mu_{g(z_p)}$, $\sigma_{g(z_p)}$ and the complexity of the problems where the parameters for the mean and the standard deviation are estimated on a random sample of 1,000,000 from *A* and the complexity derived thereof. Instead of following the original order of the problems, Table 2 gives them in ascending order of complexity (see also Table A1). The complexity varies between 2.56 for problem P11 to 14.93 for problem P30. The first 3 are less complex compared with the last 3: it is to be expected that it will be challenging to obtain a useful sample for the estimators in the most complex problems. All estimators require an efficient heuristic that produces random values in the very tail of the distribution $g(z_p)$.

Table 2: Description of the 6 problems of the OR-library.

Problem	θ	$\mu_{g(z_p)}$	$\sigma_{g(z_p)}$	Complexity
P11	7696	10760	1195	2.56
P2	4093	6054	506	3.88
P36	9934	13436	735	4.77
P13	4374	6293	276	6.95
P33	4700	6711	186	10.81
P30	1989	3335	90	14.93

In the experiments we consistently generate random values, $\tilde{x}_r^s$, in the tail by means of the Simulated Annealing (SA) heuristic (see Lenanova and Loresh, 2004). SA has been found capable to provide good solutions to the problems in the OR-library (Chiyoshi and Galvao, 2000). We believe that other heuristic methods may work equally well, but that remains to be verified as we do not consider the issue further in this paper.

For each problem we run the SA for 10,000 iterations, i.e. $R = 10{,}000$, and repeat the runs with new random start values each 100 times. Below we let $S = 10$ in the estimation process.[1] In the experiments we use the set of 100 values on $\tilde{x}_{10000}^s$ to create subsamples of $S = 10$ at random thereby approximating the outcome of an experiment (which would be very time-consuming to conduct) where obtaining S heuristic solutions to the problem were repeated a great number of times.

In the comparison of the three estimators we evaluate the following outcome variables; bias, standard deviation of the estimator, and the confidence interval's coverage percentage of the minimum θ. In Table 3 the value of the estimators are the average of 1000 repeated calculations of the estimator where 10 values of $\tilde{x}_{10000}^s$ are drawn at random each time with replacement from the

[1] We examined the length of the confidence interval obtained by the JK approach when *S* was varied from 3 up to 100. The shortening of the interval for *S* larger than 10 was modest considering the increase in computing time.

existing 100.[2] The actual minimum is subtracted from the value of the estimator to get the bias. The Bootstrap-estimator of the standard deviation for the JK-estimator is obtained by re-sampling the 10 values of $\tilde{x}^s_{10000}$, and then compute its mean over the 1000 repeats. The last column gives the average standard deviation of the 10 values of $\tilde{x}^s_{10000}$ over the 1000 repeats. Finally, the column named *SR* gives the ratio of $\sigma(\tilde{x}^s_R)/\hat{\theta}_{JK}$ which is a measure of the similarity of the heuristic solutions and thereby potentially an indication of whether the sample is in the tail near to θ or not. The results of the other 34 problems appear in Table A2 in the Appendix.

As the complexity of the problems increases, the positive bias of the JK-estimator and the EVT-estimators also increases. Up to a complexity of about 7, the bias is modest but thereafter clearly positive. Generally, the JK approach performs somewhat better than the EVT approaches in point estimating the minimum: the bias of the JK-estimator is less than or equal to the Weibull-estimator in all cases except P22 (cf Table A2).

Table 3: Average of estimator, bias and standard deviation in the computer experiments.

Problem	θ	$\hat{\theta}_{JK}$	$\hat{\theta}_{JK}-\theta$	$\sigma^*(\hat{\theta}_{JK})$	SR(‰)	$\hat{\theta}_W$	$\hat{\theta}_W-\theta$	$\sigma(\tilde{x}^s_R)$
P11	7696	7696	0	0.00	0.20	7696	0	1.58
P2	4093	4093	0	1.93	1.46	4093	0	5.97
P36	9934	9936	2	15.03	3.17	9949	15	31.46
P13	4374	4396	22	10.53	6.02	4398	24	26.41
P33	4700	4819	119	14.08	5.25	4829	129	25.32
P30	1989	2073	84	9.20	7.24	2081	92	15.01

Table 4: Confidence intervals and coverage percentages from the three estimators.

Problem	Conf. Int.^{JK}	Coverage^{JK}	Conf. Int.^{W}	Coverage^{W}	Conf. Int.^{G}	Coverage^{G}
P11	[7696,7696]	1.00	[0,7696]	1.00	[7694,7696]	1.00
P2	[4087,4093]	1.00	[31,4093]	1.00	[4089,4093]	0.93
P36	[9891,9949]	0.84	[-2856,9949]	1.00	[9940,9949]	0.32
P13	[4359,4398]	0.73	[-1417,4398]	1.00	[4388,4398]	0.11
P33	[4776,4829]	0.02	[-2352,4829]	1.00	[4826,4829]	0.00
P30	[2046,2081]	0.02	[-1093,2081]	1.00	[2079,2081]	0.00

In Table 4 the confidence intervals produced by the three estimators are given with corresponding

[2] Because the point estimates of two EVT-estimators coincide, only the Weibull estimator is shown in Table 3.

coverage proportions in repeats of 1000 (see also Table A3). In calculating the Gumbel confidence interval, the level of confidence, α, is set to 0.99. Thus, the level of confidence is at least 0.99 for all the three estimators.

From Table 4, the coverage percentage decreases from 1 to nearly 0 as the complexity increases, with the Weibull approach as an exception. The confidence intervals for the Weibull estimators consistently have as lower bound a negative value or one close to 0, which does not provide any useful information as the solutions to the problems cannot be negative. In some cases the starting values could not be calculated with the method introduced by Wilson, King and Wilson (2004), thus the interval could not be calculated as suggested by them. The Gumbel approach provide reasonable confidence intervals, but not as good as the JK approach. Hence, Table 4 gives the same conclusion as Table 3 which is that the complexity affects the performance of the estimators and the JK-estimator tend to have the best performance.

It is reasonable to infer that $\tilde{x}_R^s$ will converge when the solutions are close to the optimum, and consequently the standard deviation of them would also decrease. By dividing the standard deviation of $\tilde{x}_R^s$ by $\hat{\theta}_{JK}$ we get a measure of similarity amongst the solutions and thereby how far out in the tail close to the optimum the solutions ought to be (reported in Table 3 as column SR). The first three problems all have SR less than 5‰. Correspondingly, the bias for JK-estimator is zero or close it, and the coverage percentage is near to 1. For the last three problems being more complex SR is greater than 5‰, the bias is large, and the coverage percentage is poor. Hence, a larger SR indicates $\tilde{\mathrm{x}}_{\mathrm{r}}^{\mathrm{s}}$ are not sufficiently near to the optimum, and therefore additional iterations of the heuristic is required.

We proceed by increasing the iterations for problems P13 to P30 having SR greater than 5‰. The 10 best solutions are chosen from the 100 heuristic solutions, and the corresponding processes are continued for another 90,000 iterations. These 10 solutions, $\tilde{x}_{100000}^s$, are then used for calculating $\hat{\theta}_{JK}$ and for estimating $\sigma^*(\hat{\theta}_{JK})$ and the confidence interval of $\hat{\theta}_{JK}$ is done with the same procedure as for the results in Table 3-4. The results for the three problems are given in Table 5 (and the remaining in Table A4, Appendix).

Table 5: Results for the JK-estimator after 100,000 iterations on the more complex problems.

Problem	Dev.Times	$\hat{\theta}_{JK}$	$\hat{\theta}_{JK}-\theta$	$\sigma(\tilde{x}_R^s)$	$\sigma^*(\hat{\theta}_{JK})$	SR‰	Conf.Int	Cover
P13	6.95	4374	0	3.61	0.00	0.83	[4374,4374]	Y
P33	10.81	4709	9	10.41	3.72	2.21	[4698,4713]	Y
P30	14.93	2023	34	10.32	1.30	5.10	[2019,2023]	N

Refering to Table 5, P13 and P33 have a value of SR clearly below 5‰ after the heuristic has run for some more time. As a result, $\hat{\theta}_{JK}$ has a small bias and the 99% confidence interval covers the minimum. The large number of iterations is, on the other hand, not sufficient for the problem P30 as SR remains above 5‰ with an accompanying bias and a confidence interval which fails to cover the actual minimum. Hence, deeming from the results presented in Tables 3-5 (and in Appendix) it seems that the JK-estimator is to be preferred and that the estimator and corresponding confidence interval is to be trusted once SR is smaller than 5‰.

4. Recommendation of a stopping rule

In Section 2 and 3 we investigated the methods for estimating a minimum. Based on the findings, we suggest the following stopping rule to be used for *p*-median problems.

Step 1: Run S_1 heuristic processes with random starting values, each one with R_1 iterations.

Step 2: Calculate the variance of the S_1 solutions, the first order JK-estimator $\hat{\theta}_{JK}$ and get the ratio SR.

Step 3: If $SR < \varepsilon$, use $\hat{\theta}_{JK}$ as estimator of the minimum. Otherwise select the best S_2 processes in Step 1 and continue running until SR is smaller than ε. Thereafter calculate $\hat{\theta}_{JK}$ based on the S_2 solutions.

Step 4: Use bootstrap to obtain the standard deviation of the JK-estimator, $\sigma^*(\hat{\theta}_{JK})$, and calculate a 99% confidence interval which almost surely contains the minimum. Depending on whether the interval is sufficiently tight, stop the running of the heuristic process.

In step 1, the number of heuristic processes S_1 and iterations R_1 need to be chosen. In the experiments, setting S_1=100 and R_1=10,000 worked well on rather incomplex problems, and the variance of heuristic solutions approached zero. However, that also suggests more iterations than necessary were run and time wasted. Hence, the choice of the iterations ought to be smaller in practice. In step 3, the ε could be set to 2‰-3‰, and should not be larger than 5‰ to ensure that the estimation procedure is valid. A smaller ε would lead to higher precision in the estimates but at the cost of more computational effort.

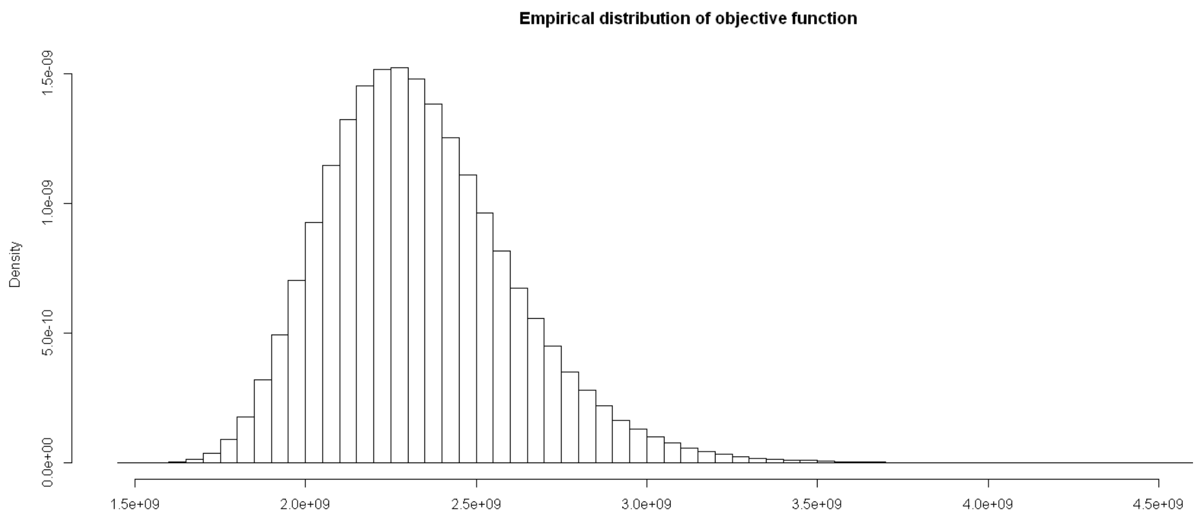
5. Illustrative problems

In this section, we first illustrate the recommended stopping rule on 3 of the problems in the OR-lib, and thereafter apply it to a practical location problem concerning allocating several distribution centers of the Swedish Post in one region. The problems of the OR-lib are P11, P13 and P37. The following settings are used: in Step 1, set S_1=100, and R_1=1,000. If SR exceeds 3‰ in step 2, set set S_2=10, and R_2=9,000 in step 3. The bootstrapping settings are the same as those in Section 3. For P11, we get SR≈0.49‰ in step 2, $\hat{\theta}_{JK} = 7696$, and $\sigma^*(\hat{\theta}_{JK}) = 0$. Furthermore, the 99% confidence interval is [7696, 7696], which indicates that the minimum is reached. For problem P13, we first get SR≈6.29‰ in step 2 and therefore increase the iterations to get in the next round; SR≈0.82‰, $\hat{\theta}_{JK} = 4374$, and $\sigma^*(\hat{\theta}_{JK}) = 0$. The 99% confidence interval is [4374, 4374], thus the minimum is found. For problem P37, SR in step 2 is 5.58‰, and decreases to 1.17‰ in step 3. In step 4, we get $\hat{\theta}_{JK} = 5071$, $\sigma^*(\hat{\theta}_{JK}) = 10.27$, and the 99% confidence interval is [5040, 5082]. At this point, the heuristic is terminated as the interval is sufficiently tight.

Finally the stopping rule is used on a real location problem in which case the minimum is unknown. The problem is to allocate 71 distribution centers of the Swedish Post on some of the 6,735 candidate nodes in the network of Dalarna in mid-Sweden. The landscape of the region and its population of 277,000 inhabitants, distributed in 15729 demand points, is described by Carling, Han, and Håkansson (2012). Han, Håkansson, and Rebreyend (2013) provides a detailed description of the road network and urge that network distance is used as distance measure. So we take the objective function as the total distance on the road network to the nearest postal center for the population.

We started with 100 processes running 10,000 iterations. In step 2, we find SR≈13.82‰ and extend the iterations by setting S_2=10 and R_2=90,000, and we get SR≈6.73‰. This ratio is still too large so additional iterations are required. We increase the iterations by adding additional 100,000 iterations and than again, and we get SR to be 5.72‰ and 4.14‰ respectively. Upon finding SR equal to 4.14‰ we decided it sufficient for estimating the minimum. In step 4, we calculated the JK-estimator to be $\hat{\theta}_{JK} = 816{,}850{,}875$, $\sigma^*(\hat{\theta}_{JK}) = 5{,}264{,}094$, and the 99% confidence interval [801058592, 822454023]. On average this means that the JK-estimator gives a minimum of 2,941 meters with a 99% confidence interval of [2888, 2961] meters. Since the confidence interval is quite tight of 70 meters we see no need for seeking an even better solution as it would only be trivially better than the current solution.

To appreciate the complexity of this real-world problem, we provide the distribution of the objective function. As we did for the OR-lib problems, we draw a random sample of 1,000,000 and show the empirical distribution of $g(z_p)$ in Figure 2. The distribution is slightly skewed to the right, but still approximately normal. To evaluate the complexity of the problem, we use $\hat{\theta}_{JK}$ as the estimate of the minimum for $g(z_p)$, and the mean and variance of $g(z_p)$ are derived by the 1 million random sample. The complexity is 5.47 which is in the middle of the 40 OR-lib problems and about 300,000 iterations were required to for SR to be below 5‰.



.

Figure 2: Empirical distribution of objective function for illustrative problem.

6. Conclusions

We have considered the problem of knowing when a solution provided by a heuristic is close to optimal. We have found the JK-estimator to successfully estimating the optimum given the existence of a sample in the tail of the distribution of the objective function. Such sample can be obtained by using the heuristic repeatedly with different, random starting values.

We have been able to suggest a stopping rule which is practical. It works for all of the OR-lib problems that represents a substantial variation in complexity as well as for a real-world problem. We consistently used Simulated Annealing as the heuristic. We do not think this choice is crucial for our findings since the heuristic only serves to obtain a sample in the tail of the distribution and any heuristic meeting this requirement should work.

We have limited the study to location problems be means of the *p*-median problems with $g(z_p)$ being the average or total distance of the population to its nearest facility. We found $g(z_p)$ to follow approximately the normal distribution. Other combinatorial problems may imply objective functions of a more complicated kind such as multi-modal or skewed distributions. We are uncertain on how the stopping rule would work under such circumstances and, hence, further investigations are required along such lines.

The stopping rule requires the heuristic to be halted and the SR to be evaluated every so often. We thought that it might be possible to forecast the number of iterations required till a valid sample in the tail was obtained. We did some attempts but the approach was not successful.

Finally, we note that an alternative strategy to the one presented here is to seek deterministic bounds. Techniques such as Lagrangian Relaxation (see Beasley, 1993) could be employed for deciding a deterministic lower bound of $\tilde{x}_R^s$, which could also be deemed as lower bound for $\hat{\theta}_{JK}$. However, that deterministic lower bound only gives the confident interval at level 1, thus its flexibility is limited. Also, the deterministic lower bound depends on the chosen parameters; therefore it is not unique for a given problem.

References

Beasley, J.E., (1990), OR library: Distributing test problems by electronic mail, Journal of Operational Research Society, 41:11, 1067-1072.

Carling, K., Han, M., Håkansson, J., (2012). Does Euclidean distance work well when the *p*-median model is applied in rural areas? Annals of Operations Research, 201:1, 83-97.

Beasley, J.E., (1993). Lagrangian heuristics for location problems. European Journal of Operational Research, 65, 383-399.

Dannenbring, D.G., (1977), Procedures for estimating optimal solution values for large combinatorial problems, Management science, 23:12, 1273-1283.

Daskin, M.S., (1995). Network and discrete location: models, algorithms, and applications. New York: Wiley.

Derigs, U, (1985). Using confidence limits for the global optimum in combinatorial optimization. Operations research, 33:5, 1024-1049.

Efron, B., (1979). Bootstrap methods: Another look at the Jackknife. Annals of statistics, 7:1, 1-26.

Golden, B.L., Alt, F.B., 1979. Interval estimation of a global optimum for large combinatorial optimization, Operations Research 33:5, 1024-1049.

Hakimi, S.L., (1964). Optimum locations of switching centers and the absolute centers and medians of a graph, Operations Research, 12:3, 450-459.

Hakimi, S.L., (1965). Optimum Distribution of Switching Centers in a Communication Network and Some Related Graph Theoretic Problems, Operations Research, 13:3, 462-475.

Han, M., Håkansson, J., and Rebreyend, P. (2013). How do different densities in a network affect the optimal location of service centers?, Working papers in transport, tourism, information technology and microdata analysis, 2013:15.

Handler, G.Y., and Mirchandani, P.B., (1979). Location on networks: Theorem and algorithms, MIT Press, Cambridge, MA.

Kotz, S., and Nadarajah, S., (2000). Extreme value distributions, theory and applications. Imperial College Press.

Levanova, T., and Loresh, M.A., (2004). Algorithm of ant system and simulated annealing for the p-median problem, Automation and Remote Control 65, 431-438.

McRobert, K.L., 1971. A search model for evaluating combinatorially explosive problems, Operations Research, 19, 1331-1349.

Nydick JR, R.L., and Weiss, H.J., (1988), A computational evaluation of optimal solution value estimation procedures, Computers & Operations Research, 5, 427-440.

Quenouille, M.H., (1956), Notes on bias in estimation, Biometrika, 43, 353-360.

Robson, D.S., and Whitlock, J.H., (1964), Estimation of a truncation point, Biometrika, 51, 33-39.

Wilson, A.D., King, R.E., and Wilson, J.R., (2004), Case study on statistically estimating minimum makespan for flow line scheduling problems, European Journal of Operational Research, 155, 439-454.

Appendix

Table A1: Description of the 40 problems of the OR-library.

Problem	θ	$\mu_{g(z_p)}$	$\sigma_{g(z_p)}$	Complexity
P1	5819	8426	877	2.97
P16	8162	11353	1033	3.09
P6	7824	10522	869	3.10
P26	9917	13644	1133	3.29
P21	9138	12906	1067	3.52
P38	11060	15078	1143	3.52
P31	10086	13960	1077	3.60
P35	10400	14179	1085	3.81
P7	5631	7930	598	3.84
P3	4250	6194	500	3.89
P27	8307	11428	727	4.29
P17	6999	9819	631	4.47
P22	8579	11699	676	4.62
P12	6634	9387	586	4.70
P39	9423	12988	736	4.84
P32	9297	12687	699	4.85
P4	3034	4618	320	4.95
P5	1355	2376	197	5.18
P8	4445	6604	356	6.07
P9	2734	4250	202	7.51
P18	4809	6769	248	7.92
P10	1255	2278	127	8.02
P23	4619	6586	220	8.94
P14	2968	4501	168	9.12
P28	4498	6369	188	9.95
P19	2845	4327	144	10.32
P15	1729	2896	109	10.67
P24	2961	4486	134	11.42
P37	5057	7246	188	11.65
P20	1789	3108	112	11.73
P40	5128	7329	179	12.32
P29	3033	4559	118	12.93
P25	1828	3131	95	13.64
P34	3013	4617	112	14.36

Table A2: Average of estimator, bias and standard deviation in the computer experiments.

Problem	θ	$\hat{\theta}_{JK}$	$\hat{\theta}_{JK}-\theta$	$\sigma^*(\hat{\theta}_{JK})$	SR(‰)	$\hat{\theta}_W$	$\hat{\theta}_W-\theta$	$\sigma(\tilde{x}_R^s)$
P1	5819	5819	0	0.00	0.00	5819	0	0.00
P16	8162	8162	0	0.11	0.92	8162	0	7.47
P6	7824	7824	0	0.00	0.00	7824	0	0.00
P26	9917	9917	0	0.50	1.20	9917	0	11.89
P21	9138	9138	0	0.01	1.05	9138	0	9.64
P38	11060	11060	0	3.09	2.34	11060	0	25.89
P31	10086	10086	0	0.25	0.79	10086	0	7.97
P35	10400	10400	0	0.35	2.89	10400	0	30.07
P7	5631	5630	-1	1.67	0.97	5631	0	5.49
P3	4250	4250	0	0.00	1.38	4250	0	5.76
P27	8307	8305	-2	6.84	2.84	8309	2	23.62
P17	6999	6998	-1	3.94	3.31	7000	1	23.18
P22	8579	8576	-3	8.46	3.59	8580	1	30.82
P12	6634	6634	0	0.22	1.59	6634	0	10.57
P39	9423	9423	0	15.24	3.59	9435	12	33.79
P32	9297	9295	-2	9.52	3.71	9301	4	34.53
P4	3034	3034	0	1.18	2.29	3034	0	6.94
P5	1355	1355	0	0.57	3.74	1355	0	5.07
P8	4445	4444	-1	4.40	4.12	4447	2	18.29
P9	2734	2741	7	8.24	6.26	2747	13	17.16
P18	4809	4853	44	14.09	5.55	4865	56	26.90
P10	1255	1270	15	4.71	8.97	1274	19	11.38
P23	4619	4687	68	17.90	7.25	4702	83	33.95
P14	2968	3008	40	14.06	8.44	3019	51	25.38
P28	4498	4565	67	20.79	6.26	4584	86	28.57
P19	2845	2912	67	10.90	6.42	2921	76	18.70
P15	1729	1767	38	8.17	9.88	1773	44	17.45
P24	2961	3050	89	11.93	6.45	3059	98	19.66
P37	5057	5201	144	16.30	5.41	5213	156	28.12
P20	1789	1848	59	9.73	9.69	1856	67	17.89
P40	5128	5272	144	18.40	5.39	5289	161	28.38
P29	3033	3131	98	14.89	7.79	3143	110	24.37
P25	1828	1905	77	6.39	8.42	1911	83	16.04
P34	3013	3126	113	10.78	6.47	3136	123	20.21

Table A3: Confidence intervals and coverage percentages from the three estimators.

Problem	Conf. Int.^{JK}	Coverage^{JK}	Conf. Int.^{W}	Coverage^{W}	Conf. Int.^{G}	Coverage^{G}
P1	[5819,5819]	1.00	-	-	[5819,5819]	1.00
P16	[8162,8162]	1.00	[-1,8162]	1.00	[8154,8162]	1.00
P6	[7824,7824]	1.00	-	-	[7824,7824]	1.00
P26	[9916,9917]	1.00	[-7,9917]	1.00	[9904,9917]	1.00
P21	[9138,9138]	1.00	-	-	[9126,9138]	1.00
P38	[11050,11060]	1.00	[-13,11060]	1.00	[11039,11060]	0.99
P31	[10085,10086]	1.00	[-15,10086]	1.00	[10077,10086]	1.00
P35	[10399,10400]	1.00	[0,10400]	1.00	[10366,10400]	1.00
P7	[5626,5631]	1.00	[-395,5631]	1.00	[5628,5631]	0.95
P3	[4250,4250]	1.00	-	-	[4243,4250]	1.00
P27	[8285,8309]	0.98	[-934,8309]	1.00	[8297,8309]	0.86
P17	[6986,7000]	0.98	[-415,7000]	1.00	[6982,7000]	0.97
P22	[8551,8580]	0.98	[-822,8580]	1.00	[8563,8580]	0.93
P12	[6633,6634]	1.00	[-1,6634]	1.00	[6623,6634]	1.00
P39	[9378,9435]	0.90	[-2197,9435]	1.00	[9426,9435]	0.43
P32	[9267,9301]	0.98	[-10073,9301]	1.00	[9283,9301]	0.89
P4	[3030,3034]	1.00	[-76,3034]	1.00	[3029,3034]	0.99
P5	[1353,1355]	1.00	[-46,1355]	1.00	[1351,1355]	1.00
P8	[4431,4447]	0.97	[-521,4447]	1.00	[4436,4447]	0.93
P9	[2716,2747]	0.81	[-913,2747]	1.00	[2743,2747]	0.11
P18	[4810,4865]	0.43	[-1986,4865]	1.00	[4859,4865]	0.00
P10	[1256,1274]	0.28	[-421,1274]	1.00	[1270,1274]	0.00
P23	[4633,4702]	0.34	[-2301,4701]	1.00	[4695,4702]	0.00
P14	[2066,3019]	0.48	[-1018,3019]	0.98	[3015,3019]	0.00
P28	[4503,4584]	0.42	[-2297,4584]	0.98	[4584,4584]	0.00
P19	[2880,2921]	0.12	[-1334,2921]	0.99	[2919,2921]	0.00
P15	[1742,1773]	0.23	[-574,1773]	1.00	[1769,1773]	0.00
P24	[3014,3059]	0.05	[-840.85,3059]	0.98	[3058,3059]	0.00
P37	[5152,5213]	0.03	[-1854,5213]	0.99	[5210,5213]	0.00
P20	[1819,1856]	0.15	[-845,1856]	1.00	[1854,1856]	0.00
P40	[5216,5289]	0.11	[-948,5289]	0.98	[5286,5289]	0.00
P29	[3086,3143]	0.10	[-1536,3143]	0.99	[3141,3143]	0.00
P25	[1887,1911]	0.01	[-522,1910]	1.00	[1906,1911]	0.00
P34	[3094,3136]	0.06	[-1558,3136]	0.99	[3132,3136]	0.00

* “-” denote the confidence interval could not be derived.

Table A4. Results for the JK-estimator after 100,000 iterations on the more complex problems.

Problem	Dev.Times	$\hat{\theta}_{JK}$	$\hat{\theta}_{JK} - \theta$	$\sigma(\tilde{x}_R^s)$	$\sigma^*(\hat{\theta}_{JK})$	SR‰	Conf.Int	Cover
P9	7.51	2736	2	7.30	3.76	2.67	[2727,2735]	Y
P18	7.92	4811	2	1.55	0.48	0.32	[4810,4811]	N
P10	8.02	1255	0	4.75	0.37	3.78	[1254,1255]	Y
P23	8.94	4621	2	9.72	2.08	2.10	[4615,4623]	Y
P14	9.12	2967	-1	6.43	0.96	2.17	[2964,2968]	Y
P28	9.95	4499	1	11.46	5.89	2.55	[4481,4500]	Y
P19	10.32	2848	3	6.83	1.10	2.40	[2845,2849]	Y
P15	10.67	1730	1	5.15	1.45	2.98	[1726,1731]	Y
P24	11.42	2964	3	10.17	7.73	3.43	[2941,2973]	Y
P37	11.65	5071	14	7.47	10.27	1.47	[5040,5082]	Y
P20	11.73	1787	-2	9.29	5.34	5.20	[1771,1793]	Y
P40	12.32	5169	41	9.33	3.81	1.80	[5158,5171]	N
P29	12.93	3056	23	10.52	4.74	3.44	[3042,3057]	N
P25	13.64	1847	19	5.52	4.51	2.99	[1833,1850]	N
P34	14.36	3056	43	8.47	3.49	2.77	[3046,3057]	N