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MODIFIED FIXED EFFECTS ESTIMATION OF TECHNICAL INEFFICIENCY

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Abstract. Abstract. We consider a method-of-moments fixed effects (FE) estimator of technical inefficiency. When dealing with a large number of cross-sectional observations, N, it is possible to obtain consistent central moments of the population distribution of the inefficiencies. It is well known that the classical FE estimator may be seriously upward biased when N is large and T, the number of time observations, is small. Based on the second central moment and assuming a single-parameter distribution of inefficiencies, we obtain unbiased technical inefficiencies in large-N settings. The proposed methodology bridges classical FE and maximum likelihood estimation, leading to a reduction in bias without making an assumption about random effects.

Key words and phrases. Technical output (in-) efficiency, Fixed effects estimator, Panel data, Method of moments (*JEL Classification:* C13, C23, C23).

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1. Introduction

It is well known that, due to random error, estimations of technical inefficiency based on the fixed effects estimator can be seriously upward biased when T, the number of time observations is small and N, the number of cross-sectional observations, is large (Kim et al. 2007, Wang & Schmidt 2009, Satchachai & Schmidt 2010).

We propose fixed effects estimators that are asymptotically unbiased as $N \to \infty$, at the expense of making distributional assumptions about the inefficiencies.

The proposed estimators are method-of-moments estimators. We consider that the inefficiencies have single-parameter distributions, e.g. corresponding to a half-normal or an exponential distribution model. It is straightforward to estimate the variance of the inefficiencies since it is equal to the variance of the firm effects. Based on an estimator of the variance of the firm effects, the type of single-parameter distribution can be identified. The parameter is estimated consistently with increasing N. Given the estimator of the single parameter, the average of the inefficiencies is obtained consistently as well as the intercept, which is then used to generate the firm-specific inefficiency estimates.

The proposed methodology is similar to the modified ordinary least squares (MOLS) estimator for cross-sectional data, but is more general in two ways. Firstly, the estimators presented here do not require distributional assumptions about the random error (not even symmetry) and, secondly, it is unnecessary to make the random effects assumption. Despite these differences, we refer to the proposed class of estimators as modified fixed effects (MFE) estimators.

MFE is not only more general than MOLS, but it also offers other advantages, e.g. MFE relies solely on the moments of the inefficiency term and not on the complete composed error term. Because of this, models with single-parameter distributed inefficiencies, such as the half-normal or the exponential model, are identified with information associated with the second-order central moment of the inefficiencies. This is a major advantage compared to MOLS, since the latter requires second-order as well as third-order information of the composed random error. Greene (2008) emphasizes the potential problem that could arise if third-order moments showed the opposite sign in the sample compared to the population. This might lead to some puzzling results, and there would be no way of determining whether the results are due to a non-representative sample, misspecification or the population having an unexpected shape. Nevertheless, this problem is avoided since MFE estimators rely solely on the variance of the estimated firm effects.

Despite the poor small-sample properties, the FE estimator may be the only option if the random effects assumption is rejected. In such cases, MFE estimators bridge the gap between Maximum Likelihood (ML) and FE estimators. For MFE estimation, the random effects assumption is not necessary and the estimator is unbiased in large-N settings given standard assumptions for ML estimators. The structure of the paper is as follows: Section 2 presents the stochastic frontier model and the MFE methodology, including both model estimation and inference. Section 3 reports a small empirical example, while Section 4 gives the conclusion.

2. The stochastic frontier model and estimation

In this study, we consider the standard linear stochastic frontier panel data model of Schmidt & Sickles (1984)

$$y_{it} = \alpha + x_{it}'\beta - u_i + \nu_{it} \equiv x_{it}'\beta + \alpha_i + \nu_{it}; \quad i = 1,\dots,N, t = 1,\dots,T, \tag{1}$$

where y_{it} is a single output, x_{it} is a $K \times 1$ vector of inputs, β is a $K \times 1$ coefficient vector and $\alpha_i = \alpha - u_i$ is the firm effect of firm i with α as the frontier intercept and u_i (≥ 0) is the measure of technical inefficiency of firm i and ν_{it} is a random error for which $E(\nu_{it}) = 0$.

The classical fixed effects estimator of the firm effect of firm i is given as follows:

$$\hat{\alpha}_i = \hat{E}(y_{it} - x_{it}'\beta|\alpha_i) = \frac{1}{T}\sum_t^T (y_{it} - x_{it}'\hat{\beta}) = \bar{y}_i - \bar{x}_i'\hat{\beta},$$

Throughout this paper, we use the within estimator of β (e.g. Wooldridge 2010). A consistent estimator of the variance of the firm effects is given by

$$\hat{\sigma}_\alpha^2 = \frac{\sum_{i=1}^N (\hat{\alpha}_i - \hat{\mu}_\alpha)^2}{N} - \frac{\hat{\sigma}_\nu^2}{T}$$

where $\hat{\mu}_\alpha = \frac{\sum_{i=1}^N \hat{\alpha}_i}{N}$ and $\hat{\sigma}_\nu^2 = \frac{\sum_i \sum_t \left[\ddot{y}_{it} - \ddot{x}_{it}'\hat{\beta}\right]^2}{N(T-1)-K}$. An MFE estimator is obtained by making a distributional assumption of the inefficiencies. In this study, we propose estimators for single-parameter (θ) distributions, for which the following reciprocal exists:

$$\hat{\theta} = \mu_2^{-1}(\hat{\sigma}_\alpha^2)$$

where μ_2 is the second central moment of the presumed single-parameter distribution. For example, if we assume the half-normal distribution, then an estimator of the single-parameter $\theta = \frac{\sqrt{\pi - 2}}{\sigma\sqrt{2}}$ is given as follows:

$$\hat{\theta} = \sqrt{\frac{\pi - 2}{2}}\frac{1}{\hat{\sigma}_\alpha}$$

and an estimator of the average population inefficiency can then be written as:

$$\hat{\mu}_u = \sqrt{\frac{2}{\pi - 2}}\hat{\sigma}_\alpha. \tag{2}$$

A consistent estimator of the intercept α, with increasing N, is obtained as follows:

$$\hat{\alpha} = \hat{\mu}_\alpha + \hat{\mu}_u \tag{3}$$

where $\hat{\mu}_\alpha = \frac{\sum_{i=1}^N \hat{\alpha}_i}{N}$. The consistency of (2) and of (3) can easily be verified with the results given in Appendix A. The inefficiencies are obtained by the classical FE approach:

$$\hat{u}_i = \hat{\alpha} - \hat{\alpha}_i$$

and since $\hat{\alpha}_i$ is an unbiased estimator of α_i, $\hat{u}_i$ asymptotically unbiased with increasing N.

Theorem 1. *Given model* (1) *and the assumptions* (i)-(v) *in Appendix B:*

$$\sqrt{N}\left(\hat{\sigma}_\alpha^2 - \sigma_\alpha^2\right) \xrightarrow{\mathfrak{L}} N(0, \dot{g}\Sigma\dot{g}')$$

as $N \to \infty$.[1] *Appendix B gives a proof and the components of the variance, $\dot{g}$ and Σ, as well as how to obtain a consistent estimator $\dot{g}\Sigma\dot{g}'$.*

Given the first order derivative of $\mu_2^{-1}(\cdot)$ exists it follows directly from Theorem 1 that

$$\sqrt{N}\left(\hat{\theta} - \theta\right) \xrightarrow{\mathfrak{L}} \frac{d\mu_2^{-1}(\sigma_\alpha^2)}{d\sigma_\alpha^2} N(0, \dot{g}\Sigma\dot{g}').$$

The proof of Theorem 1 involves showing asymptotic normality of $\sqrt{N}(\hat{\sigma}_\nu^2 - \sigma_\nu^2)$. Furthermore, in Appendix B we show asymptotic normality for the ratio

$$\frac{\hat{\sigma}_\alpha^2}{\hat{\sigma}_\alpha^2 + \hat{\sigma}_\nu^2}$$

by combining the normality of $\sqrt{N}(\hat{\sigma}_\nu^2 - \sigma_\nu^2)$ with the normality given in Theorem 1. . This enables inference about this relative measure of the influence of random error. Among others, Wang & Schmidt (2009) have discussed the poor small sample properties of the fixed effects estimator in terms of bias, when the influence of random error is relatively large. In this respect, this simple measure can serve as an indicator of whether the FE estimator is seriously upward biased.

3. Empirical example

In this section, we analyse an example based on Indonesian rice farmer data. The Indonesian Ministry of Agriculture surveyed data from six villages in West Java (Erwidodo 1990). The data were obtained from a balanced panel of 171 rice farmers over six growing seasons (three wet and three dry). Output is measured in kilograms of rice produced, and inputs are seed (kg), urea (kg), trisodium phosphate (kg), labour (hours) and land (hectares). We assume the commonly used Cobb-Douglas production function. Thus, the inputs and the output are log-transformed.

We first estimate the production function coefficients and perform a Hausman test for random effects. We also include an estimate of $\varepsilon = \frac{\sigma_u^2}{\sigma_u^2+\sigma_\nu^2}$ together with a percentile bootstrapped 95 % confidence interval. The results are given in Table 1. All coefficients are significant and the sum indicates constant returns to scale. The Hausman test rejects the null, i.e. rejects the hypothesis of random effects. Thus, there is statistical evidence that random effects estimators are inappropriate to use on this data. The estimate of the measure of the influence of random error is small $\hat{\varepsilon} = 0.134$ and the upper limit of the confidence interval is 0.218.

Hence, there is strong evidence that random error is influential and the random effects assumption is not valid. In this case, ML estimators based on the random effects assumption are inappropriate and the classical FE estimator also seems to be a poor choice. The FE estimator does not rely on the random effects assumption but likely suffers from poor finite sample properties in this particular case, when N is large, T is small and $\hat{\sigma}_u^2$ is small compared to $\hat{\sigma}_\nu^2$.

Table 2 summarizes the statistics of the FE estimator and of the two MFE estimators based on assumptions of half-normal ('HN-MFE') and exponential distributed ('E-MFE') inefficiencies. We note that a few estimates are negative for the two MFE

[1]Where, $\xrightarrow{\mathfrak{L}}$ stands for 'convergence in law', also called 'convergence in distribution'.

Variable	Coefficients	P-values
Seed	0.12	0.0001
Urea	0.10	< 0.0001
Tri. phos.	0.10	< 0.0001
Labor	0.26	< 0.0001
Land	0.44	< 0.0001
Hausman test	χ^2_5=26.7	< 0.0001
$\varepsilon = \frac{\sigma_u^2}{\sigma_u^2+\sigma_\nu^2}$	$\hat{\varepsilon} = 0.134$	95 % CI $[0.067, 0.218]$

Table 1. The within estimation results of the production function

	FE	HN-MFE	E-MFE
Mean	0.60	0.17	0.13
Std. dev.	0.19	0.19	0.19
Min.	0.00	-0.43	-0.47
1st Qu.	0.49	0.06	0.02
Median	0.61	0.18	0.15
3rd Qu.	0.70	0.28	0.23
Max.	1.03	0.60	0.56

Table 2. Summary Statistics for estimated inefficiencies u_i

estimators.This is a drawback of the MFE estimators that is found not only with MOLS estimators but also with bootstrap and jackknife bias reduction estimators (for the latter two types of estimators, see Satchachai & Schmidt 2010, and references therein). Thus, regarded as a bias reduction method, the MFE method does not differ in this respect from other proposed methods. Looking at the averages, it is clear that the distributional assumption of the MFE estimators has considerably less impact than the choice between FE and MFE estimation. Average inefficiency, as expected, is considerably higher for the FE estimator, 0.6 compared to 0.17 and 0.13 for the two MFE estimators. Thus, the results indicate exactly what we expect from classical FE estimation when N is large, and T and ε are small, that is to say, an upward bias.

When used as a tool for policymakers, the MFE estimators give an entirely different picture of the inefficiency of Indonesian rice farmers. The situation might not be as bad as the classical FE estimator indicates.

4. Conclusion

We propose a method for modified FE estimation and provide an empirical example of how this type of approach can fill a gap between classical FE and ML estimation techniques. We also propose a measure of the relative influence of random error. This is crucial in determining the small-sample behaviour of the FE estimator and, therefore, also provides information about when MFE estimation should replace classical FE estimation. There are several directions for future research on this topic. For example, making the proposed methodology robust to heteroskedastic and/or dependent random errors or changing the current setting to more complex models based on distributions with more than a single parameter.

5. Appendix A

This appendix includes various raw moment results which are needed to derive consistency and asymptotic normality for the estimators proposed in this study. We make the following assumptions:

(i) $\{x_{it}: \; t = 1, \ldots, T, \alpha_i\}$ is an iid sequence,
(ii) $\{\nu_{it}\}$ is an iid sequence,
(iii) $E(x_{itk}^4) < \infty$, $E(\nu_{it}^4) < \infty$ and $E(\alpha_i^4) < \infty$, for all $k = 1, \ldots, K$ and $t = 1, \ldots, T$,
(iv) $\sqrt{N}(\beta - \hat{\beta}) \xrightarrow{\mathfrak{L}} N\left(\mathbf{0}, \Sigma_{\hat{\beta}}\right)$
(v) $(\beta - \hat{\beta}) \xrightarrow{p} O_p(N^{-1/2})$.

Assumptions (i) and (ii) imply that the cross-section of firm is selected randomly. Assumption (ii) could be relaxed to allow for heteroskedasticity and auto-correlated errors. The usual mean independence assumption for FE estimation is strengthened by (ii). Note that both (ii) and (ii) are standard assumptions for ML estimators. The moment assumptions in (iii) are necessary for using the (strong) law of large numbers and the central limit theorem when deriving the asymptotic results.

We also need an estimator of the slope coefficients which is asymptotically normal and consistent, such as, for example, the within estimator; this is provided by assumptions (iv) and (v).

In addition to the above assumptions, two lemmas greatly simplify the proofs given in Appendix A and B a lot.

Lemma 1. *If $E|a_{it}|^4 < \infty$ for all t, then all corresponding raw moments of $\bar{a}_i$ up to $E|\bar{a}_i|^4$ are finite as well as all raw moments of $|\ddot{a}_{it}|$ up to $E|\ddot{a}_{it}|^4$ for all t.*

Proof.

$$E|\bar{a}_i|^4 \leq \frac{E\left(\sum_t |a_{it}|\right)^4}{T^4} \leq \frac{\left(\sum_t E|a_{it}|^4\right)}{T^4},$$

which is finite if all $E(a_{it}^4) \leq \infty$. The first inequality is given by the triangle inequality and the second inequality is given by Jensen's inequality. $E|\bar{a}_i|^4 < \infty$ also implies that all the lower raw moments are finite as well, e.g. $E|\bar{a}_i|^3 = E(|\bar{a}_i|^4)^{3/4} \leq (E|\bar{a}_i|^4)^{3/4}$ (by Jensen's inequality) .

Furthermore, by the triangle inequality:

$$E|\ddot{a}_{it}|^4 = E|a_{it} - \bar{a}_i|^4 \leq E|a_{it}^4| + E|-4a_{it}\bar{a}_i^3| + E|6a_{it}^2\bar{a}_i^2| + E|-4a_{it}^3\bar{a}_i| + E|\bar{a}_i^4|.$$

Each term on the right hand side should be finite. By applying the Cauchy-Schwarz inequality, we obtain:

$$E|a_{it}^2\bar{a}_i^2| \leq (E(a_{it}^4))^{1/2}(E(\bar{a}_i^4))^{1/2},$$

$$E|a_{it}\bar{a}_i^3| \leq \left(E(a_{it}^2\bar{a}_i^2)\right)^{1/2}\left(E(\bar{a}_i^4)\right)^{1/2}$$

and

$$E|a_{it}^3\bar{a}_i| \leq \left(E(a_{it}^2\bar{a}_i^2)\right)^{1/2}\left(E(a_{it}^4)\right)^{1/2}$$

which are all finite if $E(a_{it}^4) < \infty$ for all t. □

Lemma 2. *Given assumption* (v) *and* $E(|a_{ik}^p|) < \infty$ *for all* k, *then*

$$\frac{\sum_{i=1}^{N}\left(a_i'(\beta-\hat{\beta})\right)^p}{N} = O_p\left(N^{-p/2}\right)$$

for any finite $p \geq 1$.

Proof.

$$\frac{\sum_{i=1}^{N}\left(a_i'(\beta-\hat{\beta})\right)^p}{N} = \frac{\sum_{i=1}^{N}\left(\sum_{k=1}^{K} a_{ik}(\beta_k-\hat{\beta}_k)\right)^p}{N} \leq$$

$$\frac{\sum_{i=1}^{N}\left(\sum_{k=1}^{K} a_{ik}^2 \sum_{k=1}^{K}(\beta_k-\hat{\beta}_k)^2\right)^{p/2}}{N} \leq \left(\sum_{k=1}^{K}(\beta_k-\hat{\beta}_k)^p\right)\sum_{k=1}^{K}\left(\frac{\sum_{i=1}^{N} a_{ik}^p}{N}\right)$$

$$= O_p(N^{-p/2})O_p(1) = O_p(N^{-p/2}).$$

The first inequality is given by applying the Cauchy-Schwarz inequality and the second inequality is given by the Jensen's inequality. □

The following results for raw moments are necessary to obtain the main asymptotic results given in Appendix B.

$$\hat{E}(\nu_{it}) = 0 \tag{4}$$

$$\hat{E}(\nu_{it}^2) = \frac{\sum_i\sum_t \hat{\hat{\nu}}_{it}^2}{N(T-1)} \xrightarrow{p} E(\nu_{it}^2) = \sigma_\nu^2 \tag{5}$$

$$\hat{E}(\nu_{it}^3) = \frac{\sum_i\sum_t \hat{\hat{\nu}}_{it}^3}{N}\frac{T}{(T-1)(T-2)} \xrightarrow{p} E(\nu_{it}^3) \tag{6}$$

$$\hat{E}(\nu_{it}^4) = \frac{\sum_i\sum_t \hat{\hat{\nu}}_{it}^4}{N}\frac{T^2}{(T-1)(T^2-3T+3)} - \frac{3(2T-3)}{T^2-3T+3}\sigma_\nu^4 \xrightarrow{p} E(\nu_{it}^4) \tag{7}$$

$$E(\bar{\nu}_i) = \frac{\sum_t E(\nu_{it})}{T} = 0 \tag{8}$$

$$E(\bar{\nu}_i^2) = \frac{E(\nu_{it}^2)}{T} = \frac{\sigma_\nu^2}{T} \tag{9}$$

$$E(\bar{\nu}_i^3) = \frac{E(\nu_{it}^3)}{T^2} \tag{10}$$

$$E(\bar{\nu}_i^4) = \frac{E(\nu_{it}^4)}{T^3} + \frac{3(T-1)\sigma_\nu^4}{T^3} \tag{11}$$

To identify the parameter of the single parameter distribution of the inefficiencies (and to identify α) we need consistent estimators of the two first raw moments of the firm effects.

$$\hat{E}(\alpha_i) = \frac{\sum_i \hat{\alpha}_i}{N} \xrightarrow{p} \mu_\alpha \tag{12}$$

$$\hat{E}(\alpha_i^2) = \frac{\sum_i \hat{\alpha}_i^2}{N} - \hat{E}(\bar{\nu}^2) \xrightarrow{p} E(\alpha_i^2) \tag{13}$$

The following results for higher raw moments are useful when computing estimators for the asymptotic variances presented in Appendix B:

$$E(\alpha_i^3) = \alpha^3 - 3\alpha\mu_\alpha + 3\alpha E(\alpha_i^2) - E(u_i^3) \tag{14}$$

$$E(\alpha_i^4) = E(u_i^4) - \alpha^4 + 4\alpha^3\mu_\alpha - 6\alpha^2 E(\alpha_i^2) + 4\alpha E(\alpha_i^3). \tag{15}$$

In the following, we verify the consistency of the estimator of the fourth raw moment of ν_{it}. The results for the lower raw moments can be verified similarly.

$$\hat{E}(\nu_{it}^4) = \frac{\sum_i \sum_t \hat{\ddot{\nu}}_{it}^4}{NT} = \frac{\sum_i \sum_t (\ddot{y}_{it} - \ddot{x}_{it}'\hat{\beta})^4}{NT} = \frac{\sum_i \sum_t \left[\ddot{\nu}_{it} + \ddot{x}_{it}'(\beta - \hat{\beta})\right]^4}{NT} = \tag{16}$$

$$\frac{\sum_i \sum_t \left[\ddot{\nu}_{it}^4 + 4\ddot{\nu}_{it}^3 \ddot{x}_{it}'(\beta - \hat{\beta}) + 6\ddot{\nu}_{it}^2 \left(\ddot{x}_{it}'(\beta - \hat{\beta})\right)^2\right]}{NT} +$$

$$+ \frac{\sum_i \sum_t \left[4\ddot{\nu}_{it}\left(\ddot{x}_{it}'(\beta - \hat{\beta})\right)^3 + \left(\ddot{x}_{it}'(\beta - \hat{\beta})\right)^4\right]}{NT}$$

[Given assumptions (i), (ii), (iii) and lemmas (1) & (2)]

$$\xrightarrow{p} \frac{\sum_t E(\ddot{\nu}_{it}^4)}{T} = \frac{\sum_t E(\nu_{it} - \bar{\nu}_i)^4}{T} =$$

$$= \frac{\sum_t E(\nu_{it}^4)}{T} - 4\frac{\sum_t E(\nu_{it}^3 \bar{\nu}_i)}{T} + 6\frac{\sum_t E(\nu_{it}^2 \bar{\nu}_i^2)}{T} - 3\frac{\sum_t E(\bar{\nu}_i^4)}{T} =$$

[Given (ii)]

$$= E(\nu_{it}^4)\left[\frac{(T-1)(T^2 - 3T + 3)}{T^3}\right] + \frac{3(T-1)(2T-3)\sigma_\nu^4}{T^3}$$

It is now straightforward to verify (7). We now continue by verifying the right-hand side of (11); although the lower moments are not derived explicitly here, they can be derived by analogy in a similar way.

$$E(\bar{\nu}_i^4) = \frac{E\left(\sum_{t=1}^T \nu_{it}\right)^4}{T^4} = \tag{17}$$

$$= \frac{1}{T^4}\left(\sum_{t=1}^T E(\nu_{it}^4) + 4\sum_{t=1}^T \sum_{s \neq t} E(\nu_{it}^3 \nu_{is}) + 3\sum_{t=1}^T \sum_{s \neq t} E(\nu_{it}^2 \nu_{is}^2) + \right.$$

$$\left. +6\sum_{t=1}^T \sum_{s \neq t} \sum_{r \neq t,s} E(\nu_{it}^2 \nu_{is} \nu_{ir}) + \sum_{t=1}^T \sum_{s \neq t} \sum_{r \neq t,s} \sum_{q \neq t,s,r} E(\nu_{it}\nu_{is}\nu_{ir}\nu_{iq})\right) =$$

[Given (ii), (iii) and lemmas 1 & 2]

$$= \frac{E(\nu_{it}^4)}{T^3} + \frac{3(T-1)\sigma_\nu^4}{T^3}$$

The multinomial theorem is employed to expand the sum raised to the power four in (17). Finally $\hat{E}(\alpha_i^2)$ in (13) is verified by deriving the probability limit of $\frac{\sum_i \hat{\alpha}_i^2}{N}$.

$$\frac{\sum_i \hat{\alpha}_i^2}{N} = \frac{\sum_i \left[\bar{y}_i - \bar{x}_i'\hat{\beta}\right]^2}{N} = \frac{\sum_i \left[\alpha_i + \bar{\nu}_i + \bar{x}_i'(\beta - \hat{\beta})\right]^2}{N} = \tag{18}$$

[Given (i), (ii), (iii) and lemmas (1) & (2)]

$$\xrightarrow{p} E(\alpha_i^2) + E(\bar{\nu}_i^2).$$

6. Appendix B

In this appendix, we first give a proof of Theorem 1 and then develop a consistent estimator of the asymptotic variance of $\sqrt{N}\left\{\hat{\sigma}_\alpha^2 - \sigma_\alpha^2\right\}$ Finally, we derive the asymptotic normality of $\sqrt{N}\left[\frac{\hat{\sigma}_\alpha^2}{\hat{\sigma}_\alpha^2+\hat{\sigma}_\nu^2} - \frac{\sigma_\alpha^2}{\sigma_\alpha^2+\sigma_\nu^2}\right]$.

$$\sqrt{N}\left\{\hat{\sigma}_\nu^2 - \sigma_\nu^2\right\} = \sqrt{N}\left\{\frac{\sum_i\sum_t \hat{\ddot{\nu}}_{it}^2}{N(T-1)} - \sigma_\nu^2\right\} = \sqrt{N}\left\{\frac{\sum_i\sum_t \left[\ddot{y}_{it} - \ddot{x}_{it}'\hat{\beta}\right]^2}{N(T-1)} - \sigma_\nu^2\right\}$$

$$= \sqrt{N}\left\{\frac{\sum_i\sum_t \left[\ddot{\nu}_{it} + \ddot{x}_{it}'\left(\beta - \hat{\beta}\right)\right]^2}{N(T-1)} - \sigma_\nu^2\right\}$$

[Given (i), (ii), (iii) and lemmas 1 & 2]

$$\xrightarrow{p} \sqrt{N}\left\{\frac{\sum_t\sum_i \ddot{\nu}_{it}^2}{N(T-1)} - \sigma_\nu^2\right\}.$$

Thus,

$$\sqrt{N}\left\{\frac{\sum_i\sum_t \hat{\ddot{\nu}}_{it}^2}{N(T-1)} - \sigma_\nu^2\right\} - \sqrt{N}\left\{\frac{\sum_t\sum_i \ddot{\nu}_{it}^2}{N(T-1)} - \sigma_\nu^2\right\} \xrightarrow{p} 0$$

and, therefore, $\sqrt{N}\left\{\frac{\sum_i\sum_t \hat{\ddot{\nu}}_{it}^2}{N(T-1)} - \sigma_\nu^2\right\} - \sqrt{N}\left\{\frac{\sum_t\sum_i \ddot{\nu}_{it}^2}{N(T-1)} - \sigma_\nu^2\right\} \xrightarrow{\mathfrak{L}} 0$. And given (ii) and (iii)

$$\sqrt{N}\left\{\frac{\sum_t\sum_i \ddot{\nu}_{it}^2}{N(T-1)} - \sigma_\nu^2\right\} \xrightarrow{\mathfrak{L}} N\left(0, V\left(\frac{\sum_t \ddot{\nu}_{it}^2}{T-1}\right)\right),$$

where

$$V\left(\frac{\sum_t \ddot{\nu}_{it}^2}{T-1}\right) = \frac{T-1}{T^2}\left(E(\nu_{it})^4 - \frac{(T-3)}{(T-1)}\sigma_\nu^4\right).$$ [2]

Let us now consider the counterpart for the variance estimator of the firm effects (and for the inefficiencies):

$$\sqrt{N}\left\{\hat{\sigma}_\alpha - \sigma_\alpha^2\right\} = \sqrt{N}\left\{\frac{\sum_i \hat{\alpha}_i^2}{N} - \hat{\mu}_\alpha^2 - \frac{\hat{\sigma}_\nu^2}{T} - \sigma_\alpha^2\right\}. \tag{19}$$

[2]The variance is straightforward to derive using systematic methods for central moments (see Rao 1973, p 438).

Assumptions (i), (ii), (iii) and lemmas 1 & 2 are sufficient assumptions for (19) to be asymptotic equivalent to

$$\sqrt{N}\left\{\frac{\sum_i(\alpha_i+\bar{\nu}_i)^2}{N}+2E(\alpha_i\bar{x}_i)'(\beta-\hat{\beta})-\left[\frac{\sum_{i=1}^N\alpha_i+\bar{\nu}_i}{N}+\frac{\sum_{i=1}^N\bar{x}_i'(\beta-\hat{\beta})}{N}\right]^2-\right.$$
$$\left.\frac{\sum_i\sum_t\ddot{\nu}_{it}^2}{NT(T-1)}-\sigma_\alpha^2\right\}\equiv$$

$$\sqrt{N}\left\{\frac{\sum_i A_i}{N}+2E(\alpha_i\bar{x}_i)'B-\left[\frac{\sum_{i=1}^N C_i}{N}+\frac{\sum_{i=1}^N D_i'B}{N}\right]^2-\frac{\sum_i E_i}{N}-\sigma_\alpha^2\right\}. \tag{20}$$

Given the assumptions (i), (ii), (iii) and (iv) the Lindeberg-Lévy central limit theorem can be applied to each term in (20) such that:

$$N^{-1/2}\sum_{i=1}^N\left[\begin{pmatrix}A_i\\B\\C_i\\D_i\\E_i\end{pmatrix}-\begin{pmatrix}E(A_i)\\E(B)\\E(C_i)\\E(D_i)\\E(E_i)\end{pmatrix}\right]\xrightarrow{\mathfrak{L}}N_{2K+3}(\mathbf{0},\mathbf{\Sigma}). \tag{21}$$

Asymptotic normality for $\hat{\sigma}_\alpha^2$ is then given as

$$\sqrt{N}\left\{\hat{\sigma}_\alpha-\sigma_\alpha^2\right\}\xrightarrow{\mathfrak{L}}N\left(0,\dot{g}\mathbf{\Sigma}\dot{g}'\right)$$

where $\dot{g}=[\,1\quad 2E(\alpha_i\bar{x}_i)'-2\mu_\alpha E(\bar{x}_i)'\quad -2\mu_\alpha\quad \mathbf{0}_{1\times K}\quad -1\,]$.

This concludes the proof of Theorem 1. Given (i), (ii), (iii), (iv) and lemmas 1 & 2, we can derive the following asymptotic covariance matrix:

$$\mathbf{\Sigma}=\begin{bmatrix}V(A_i)&Cov(A_i,B)'&Cov(A_i,C_i)&Cov(A_i,D_i)'&Cov(A_i,E_i)\\Cov(A_i,B)&V(B)&Cov(B,C_i)&Cov(B,D_i)&Cov(B,E_i)\\Cov(A_i,C_i)&Cov(B,C_i)'&V(C_i)&Cov(C_i,D_i)'&Cov(C_i,E_i)\\Cov(A_i,D_i)&Cov(B,D_i)&Cov(C_i,D_i)&V(D_i)&Cov(D_i,E_i)\\Cov(A_i,E_i)&Cov(B,E_i)'&Cov(C_i,E_i)&Cov(D_i,E_i)'&V(E_i)\end{bmatrix},$$

where

$$E(A_i)=E(\alpha_i^2)+\frac{\sigma_\nu^2}{T};\;E(B)=\mathbf{0}_{K\times 1};\;E(C_i)=\mu_\alpha;\;E(D_i)=E(\bar{x}_i);\;E(E_i)=\frac{\sigma_\nu^2}{T};$$
$$V(A_i)=\left[V(\alpha_i^2)+V(\bar{\nu}_i^2)+4E(\alpha_i^2)E(\bar{\nu}_i^2)+4\mu_\alpha E(\bar{\nu}_i^3)\right];\;V(B)=\Sigma_{\hat{\beta}};$$
$$V(C_i)=\sigma_\alpha^2+E(\bar{\nu}_i^2);\;V(D_i)=E(\bar{x}_i\bar{x}_i')-E(\bar{x}_i)E(\bar{x}_i)';$$
$$V(E_i)=\frac{T-1}{T^4}\left(E(\nu_{it})^4-\frac{(T-3)}{(T-1)}\sigma_\nu^4\right);$$
$$Cov(A_i,B)=\mathbf{0}_{K\times 1};\;Cov(A_i,C_i)=\left[E(\alpha_i^3)+2\mu_\alpha E(\bar{\nu}_i^2)-\mu_\alpha E(\alpha_i^2)+E(\bar{\nu}_i^3)\right];$$
$$Cov(A_i,D_i)=E(\alpha_i^2\bar{x}_i)-E(\alpha_i^2)E(\bar{x}_i);\;Cov(A_i,E_i)=\left[E(\bar{\nu}_i^4)-\frac{3\sigma_\nu^4}{T^2}+2\mu_\alpha E(\bar{\nu}_i^3)\right];$$
$$Cov(B,C_i)=\mathbf{0}_{K\times 1};\;Cov(B,D_i)=\mathbf{0}_{K\times K};\;Cov(B,E_i)=\mathbf{0}_{K\times 1};$$
$$Cov(C_i,D_i)=E(\alpha_i\bar{x}_i)-E(\alpha_i)E(\bar{x}_i);\;Cov(C_i,E_i)=E(\bar{\nu}_i^3);\;Cov(D_i,E_i)=\mathbf{0}_{K\times 1}.$$

A consistent estimator of $\dot{g}\mathbf{\Sigma}\dot{g}'$ as $N \to \infty$ can be constructed using the results in (4)-(15) along with the consistent estimators of $E(\bar{x}_i)$, $E(\alpha_i \bar{x}_i)$, $E(\alpha_i^2 \bar{x}_i)$ and $\Sigma_{\hat{\beta}}$. For the last four terms, we can use $\sum_{i=1}^N \bar{x}_i/N$, $\sum_{i=1}^N(\hat{\alpha}_i\bar{x}_i)/N$, $\sum_{i=1}^N(\hat{\alpha}_i^2\bar{x}_i)/N - \sum_{i=1}^N \bar{x}_i/N\hat{\sigma}_\nu^2/T$ and $\hat{\sigma}_\nu^2\left(\sum_{i=1}^N\sum_{t=1}^T \ddot{x}_{it}'\ddot{x}_{it}\right)^{-1}$, which are consistent as $N \to \infty$.

The above considerations show that both $\sqrt{N}\left\{\hat{\sigma}_\nu^2 - \sigma_\nu^2\right\}$ and $\sqrt{N}\left\{\hat{\sigma}_\alpha - \sigma_\alpha^2\right\}$ are asymptotically normally distributed. We note that $\hat{\sigma}_\nu^2$ is asymptotically equivalent to $T\sum_{i=1}^N E_i/N$, so the normality result in (21) can be used to obtain the following asymptotic normality result:

$$\sqrt{N}\left(\frac{\hat{\sigma}_\alpha^2}{\hat{\sigma}_\alpha^2+\hat{\sigma}_\nu^2} - \frac{\sigma_\alpha^2}{\sigma_\alpha^2+\sigma_\nu^2}\right) \xrightarrow{\mathfrak{L}} N\left(0, \dot{h}\mathbf{\Sigma}\dot{h}'\right),$$

$$\dot{h} = \frac{\sigma_\nu^2}{(\sigma_\alpha^2+\sigma_\nu^2)^2}\left[\begin{array}{ccccc} 1 & 2E(\alpha_i\bar{x}_i) - 2\mu_\alpha E(\bar{x}_i) & -2\mu_\alpha & \mathbf{0}_{K\times 1} & -\dfrac{\sigma_\nu^2/T+\sigma_\alpha^2}{\sigma_\nu^2/T}\end{array}\right].$$

A R script with the analytical estimation of the asymptotic covariances given in Appendix B can be provided on request.

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