A reason to believe: beliefs as an influence on students task solving.

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Abstract. Upper secondary students' task solving reasoning was analysed, with a focus on what grounds they had for different strategy choices and conclusions. Beliefs were identified and connected with the reasoning that took place. The results indicate that beliefs have an impact on the central decisions made during task solving. Three themes stand out: safety, expectation and motivation.

Contents

1. Introduction

A body of research has pointed out the important role that student's beliefs play in problem solving (Schoenfeld, 1985; Lester et al., 1989; Philippou and Christou, 1998; Thompson, 1992). Consequences of beliefs could be:

- Mathematics is based on rules, which means I need to learn the rules to be able to succeed in mathematics.
- I'm able to solve problems and therefore I don't panic when I'm facing a type of problem I haven't met before.
- Teaching is telling and as a student I have nothing to add.
- Learning is competitive and you need to be strong to succeed in mathematics.

Students' way of tackling mathematical problems are constrained by their beliefs (Schoenfeld, 1992; Wong et al., 2002). Barkastas and Hunting (1996) concluded that counter-productive beliefs must be identified and be dealt with on an individual basis.

The aim of this study is to investigate what influence beliefs have in problem solving and more specifically how they affect reasoning used in solving problematic situations. Here, I wish to find out what type of beliefs are behind different reasoning, clarify the differences between them and investigate how they affect the students reasoning. To be able to do that, I will attribute beliefs to students behaviour, and as a test of accuracy see if the student behave in ways consistent with them "having" them. I can't say that people have beliefs, only that they act in ways consistent with them having those beliefs. This type of attribution is helpful in predicting and explaining behaviour.

The research questions is:

Q How do beliefs influence the central decisions students make in their reasoning while solving problematic situations?

There is also an aim to create a structure for analysing students' beliefs in relation to their mathematical reasoning.

2. Theoretical framework

2.1. Beliefs. There is no single definition of beliefs in the field of mathematics education (Furinghetti and Pehkonen, 2002) but three major concepts have been identified in the research of affect (McLeod, 1992): beliefs, attitudes and emotions. Schoenfeld (1985) describes beliefs as the perspective with which one approaches mathematics and mathematical tasks. He refines this definition by saying that beliefs could "be interpreted as an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior" (Schoenfeld, 1992) (p.358). This broad definition is accepted by a majority of researchers in the field as at least a starting point for investigations of beliefs(Furinghetti and Pehkonen, 2002).

There are good reasons for excluding emotions (or feelings) from a definition of belief. Hannula (2004) provides us with following example (p.50):

> For example, two students may share a cognitive belief that problem solving is not always straightforward, but this belief might be associated with enjoyment for one and with anxiety for the other.

Hannula (2006) uses three concepts, typically adopted by psychologists (Meyer and Turner, 2002), to describe human learning: cognition, motivation, and emotion. Under this conceptual framework, beliefs have components in both the cognitive area and the emotional area and that is why it could be difficult to separate an emotion or attitude from a belief. They also appear to be in symbiosis, such that beliefs often interact with and, at times, shape attitudes and emotions (Lester et al., 1989) or as Cobb, Yackel and Wood (1989) describe it (p.140): "emotional acts depends on beliefs". Furthermore, one's self-image or self-concept is part of beliefs (Hannula, 2004). Goldin (2004) adds one more level when he talks about meta- affect: affect about affect, affect about cognition about affect and affective context of affect (dependent on cognition).

Given these difficulties, Hannula (2006) uses three different timeframes to allow discussion about beliefs in different contexts. These are, from shortest to longest,

Rapid self-regulation: of actions and thoughts (e.g. solving a given mathematical task)

State aspect: which is an intermediate timeframe that regards self and psychological traits as stable constructs, but allowing manipulation of context (e.g. a student solving the problem may start collaborating with a peer)

Trait aspect: where psychological traits are constructed and reconstructed (e.g. the student may become more confident through a series of successful problem solving episodes)

Each of these time frames provide a different focus on the three concepts of cognition, motivation and emotions. Identifying the timeframe of interest then provides the correct focus for our definition of beliefs. Different definitions of beliefs aim toward different aspects of beliefs which produce different research questions.

The focus in this paper is on the beliefs that are activated in the process of solving problematic situations. Such situations lie in the second of Hannula's timeframes, the state aspect. This is in contrast to studying how a tendency to believe in a certain way in certain type of tasks, which would fall in the third timeframe. If the behaviour happens repeatedly during problem solving, we could consider it a tendency. While I don't want to diminish the impact of such social contexts when it comes to how beliefs are created, in the current paper I focus my research by not looking in to this side of beliefs.

With this timeframe identified I, following Hannula (2006), define

Belief: as an individual's understandings that shape the ways that the individual conceptualizes and engages in mathematical behaviour generating and appearing as thoughts in mind. In this sense beliefs are primarily cognitive. They are neither objective knowledge nor emotion. (e.g. It is easier to understand a function if you look at the graph.)

Motivation: as active goals, intrinsic or extrinsic. (e.g. I want to solve this task because I want to feel good about myself (internal cause) or I want to pass this course (external cause).)

Emotion: as an emotional state, which includes mood. (e.g. dislike)

These are the definitions used in this paper.

There are several ways to categorize mathematical beliefs (Op't Eyende et al., 2002). Schoenfeld (1985) divides beliefs in to four different categories: beliefs about mathematics (e.g. mathematics is based on rules), beliefs about self (e.g. I am able to solve problems), beliefs about mathematics teaching (e.g. teaching is telling), and beliefs about the social context (e.g. learning is competitive).

How do we identify students' beliefs? When a decision is made in a problem solving situation, there is a justification presented as argument for what is being decided. According to Schoenfeld (1985), beliefs set the context where the individuals resources, heuristics and control operates. When a student makes a specific decision (conscious or not, well grounded or not, correct or not) during a solution procedure, it has been influenced by one or several beliefs. During reasoning in a problem solving procedure following decisions will be affected: identifying problematic situations, choice of strategy, implementation of strategy and evaluation of conclusions. These will be made on grounds where the arguments can be implicit or explicit. The arguments aim to answer the question "Why did the student do this and not that?". Since the decisions are influenced by beliefs, the arguments are affected by beliefs making some arguments more valid or relevant than others.

When categorising beliefs it is important to remember that an individual's beliefs are linked to each other in a system, and the individual defines the links in this system itself. Schoenfeld (1985; 1992) describes this belief system as the individual's mathematical world view and it is often created without the person being aware of it. The notion of a belief system is a metaphor used to describe how one's beliefs are organized (Green, 1971). It consists of three dimensions: quasi- logicalness, psychological centrality and cluster structure. Each of an individual' beliefs is dependent on their other beliefs. They are connected and the relationship between different beliefs is not necessarily logical, since it is the individual herself that arranges them from how she sees these connections. There is a *quasi- logical structure* with primary beliefs and some derivative beliefs. How convinced an individual is about something depends on the *psychological strength* of the belief. A belief could be central and are strongly held, or peripheral and likely to change. This dimension doesn't exist in a knowledge system (Furinghetti and Pehkonen, 2002). If you *know* something, you are not likely to accept any contradiction to this. Beliefs are held in *clusters*. These clusters don't necessarily have any relationship to each other and therefore can be kept more or less isolated. The reasons for seeing beliefs as a part of a system is because beliefs are not isolated and they are context/ situation bound. They function in operational terms as a part of a model of cognition.

2.2. Reasoning. This study follows on from previous research on mathematical reasoning (Bergqvist et al., 2003). The same, but now extended, theoretical framework about reasoning will be used in this study. In this paper, reasoning has the same meaning as in Lithner (2003): the line of thought, the way of thinking, adopted to produce assertions and reach conclusions. In this definition, reasoning doesn't necessarily have to be based on formal deductive logic, and it may even be incorrect. The core of the framework is to describe and classify the reasoning that takes place while solving problematic situations. To organize the data, I will use a four step reasoning structure:

(1) A *problematic situation* (PS) is met where it is not obvious (for the individual) how to proceed.
(2) *Strategy choice* (SC): try to choose (in a wide sense: recall, construct) a strategy that can solve the problematic situation.
(3) *Strategy implementation* (SI).
(4) *Conclusion* (C): A result is obtained.

In step two, a predictive argumentation can support the strategy choice and in step three, a verifying argumentation can support the implementation of the strategy. Argumentation is considered being the substantiation, the part of reasoning that fills the purpose of convincing you or someone else that the reasoning is appropriate.

The next step is to classify the argumentation concerning the strategy choice and conclusion. There are two main types of reasoning: imitative (which is a family of different types of, from a mathematical point of view, superficial reasoning) and creative mathematical reasoning. Creative Reasoning (CR) is reasoning that is novel flexible, plausible and has a mathematical foundation, all which Imitative Reasoning does not require.

Following types of Imitative Reasoning is the result from previous research (Lithner, 2006):

- *Memorised Reasoning* (MR). The strategy choice is founded on recalling a memorized answer. The strategy implementation consists of writing this answer down with no other consideration.
- *Algorithmic Reasoning* (AR). Recalling a certain algorithm (set of rules) will probably solve the problematic situation. The strategy implementation is straight forward once the rules is given (recalled).

- *Familiar MR/ AR.* The strategy choice is founded on identifying the task as being familiar that can be solved by a certain algorithm or recalling a complete answer. As for strategy choice, the algorithm is implemented (AR) or the answer is recalled (MR)
- *Delimiting AR.* An algorithm is chosen from a set. The set is delimited through, in relation to the task, surface property by the reasoner. The strategy implementation is carried out by following the algorithm. If not successful, the algorithm is abandoned and a new one is chosen.
- *Guided AR.* The main strategy choice is to find external algorithmic guidance from two different sources:
  (1) Text-guided AR. The strategy choice is founded on identifying similarities (on a surface level) between the task and an object in a text source. These objects could be an example, a definition, a theorem or a rule. This identification does not rely on any intrinsic mathematical properties. The strategy implementation is to copy the procedure that has been identified, without any verifying argument.
  (2) Person-guided AR. All strategy choices that could have been problematic for the solver are made and controlled by someone else. The person who guides gives no predictive argument to support the local and global strategy choices. The strategy implementation is to follow the guidance.

A reasoning is defined as Creative Mathematically Founded Reasoning (CR) if it fulfils following conditions (Lithner, 2006):

i) Novelty: A new (to the reasoner) sequence of solution reasoning is created, or a forgotten sequence is re- created. To imitate an answer or a solution procedure is not seen as novel.

ii) Flexibility: It fluently admits different approaches and adaptations to the situations. It does not suffer from fixations that hinder the progress. Such fixation can be content universe fixation, or a fixation to search for memorised or algorithmic solutions.

iii) Plausibility: There are arguments supporting the strategy choice and/ or strategy implementation, motivating why the conclusions are true or plausible. Guesses, vague intuitions and affective reasons are not considered.

iv) Mathematical Foundation: The argumentation is founded on intrinsic mathematical properties of the component involved in the reasoning. Purely experience-based reasons as in keyword strategy are not valid. Intrinsic is considered being central (in comparison to surface properties which have no or little relevance). Mathematical is what is accepted by the mathematical society as being correct. Property is part of a component: objects (fundamental entity, e.g. numbers, functions, graphs); transformations (operations on an object, e.g. calculate a determinant); and concepts (central mathematical idea built on a related set of objects, transformations and their properties, e.g. concept of infinity).

3. Method

The study was presented to two teachers in an upper secondary school. The teachers were asked to pick out four (two girls and two boys) students in their eleventh school year (age 17 - 18) from the natural science programme (NV), the most mathematical intense in the Swedish school system, with the instruction to avoid students with extremely good or poor results. Such students should have the resources required for the tasks, but still find problematic situations in them. The same three tasks, all with different degree of difficulty, as in the earlier study (Bergqvist et al., 2003) were used. Similar ones could be found in the students' textbook. Three of the students that were asked to participate came from the natural science programme. The fourth student came from the technology programme but was reading the same mathematics courses as the other three. They were asked to participate in the study in their spare time, which they all agreed to do. Written information about the study was handed out to the headmaster, and to the teachers and students that were participating in the study. The students signed a paper where they approved the use of the data, and were encouraged to inform their parents about the study.

3.1. Forms of data collection. There are many ways of trying to capture students' beliefs (Leder and Forgasz, 2002), all with different focus on measuring beliefs. In this study, I used three different methods to collect data: observation, interview and questionnaire. The reason for using several methods was to increase the probability of grasping more of this complex subject. It can be difficult to catch beliefs with one single method. In this study, the reasoning and the beliefs that are shown in the empirical data will be investigated and analysed. There are more beliefs that affect the students reasoning which are not going to be revealed in the data. I now describe the four stages of data collection in detail.

Observation
The sessions were video taped with the camera directly above the sheet of paper, recording the students written work and the use of a calculator as well as what they were saying. The intention was that the students were supposed to think aloud and solve the task in a test- like situation with the researcher as an observer. A pilot study showed that not much was revealed about the students' beliefs mainly because the students were too quiet. Therefore in the current study a more active observer, an interviewer, asked questions if the students were silent for a longer period of time. The questions were of the kind "What are you thinking now?" and "What are you doing now?". I can't escape the influence the social context, how the data were collected, has on the students. They are aware of the camera, the interviewer, the test etc. Some of them commented on the awkwardness in the beginning of the problem solving session. The students were informed that they could stop the experiment whenever they wanted to. But, after a few minutes they accepted the situation and all the students proceed with the problem solving.

Interview 1
Each solving session was limited to 30 minutes. Immediately after the session, a short interview was made where the student could further explain their solving procedures and the way of thinking.

Questionnaire
The purposes of the questionnaire were to use it as a background for the second interview, to give a general picture of the students view, but also to help understanding some of the central decisions that the students made. It consisted of 23 items with a Likert- scale in four steps. The reason for four steps, instead of five, was to force the students to take a stand whether they agreed or not with each statement. The students were informed that if they did not have any opinion about one of the statements, they could leave it blank. All students answered all the questions. The items on the questionnaire were beliefs that have been identified in previous research (Schoenfeld, 1985; Carlson, 1999; Frank, 1988).

Interview 2
An interpretation was made with the aim to describe the problematic situations, and the students' action, reasoning and beliefs in these situations. A second semi- structured interview (about 20 minutes) took place two days after the first session where the interpretation was presented to the students. The students were asked to comment on the suggested interpretation. This interview had the purpose to clarify *why* the students acted as they did in the problematic situations they were facing, and there was a possibility to create a dialog between the researcher and the student.

3.2. Method of analysis. The goal of the analysis is to present and highlight plausible reasons for why the students acted in the way they did by looking for consistent behaviour and possible underlying causes, and attribute possible beliefs as these causes.

1. Data The data were transcribed and a description and interpretation of the problem solving sessions was made based on the video recordings and the interviews.

2. Problematic situations (PS) The problematic situations were identified and an appropriate grain size chosen in cases that the problematic situations could be sub- divided in to several problematic situations.

3. Central decisions Belief influences the student's decisions and the argumentation for these decisions during all four steps[1] of reasoning. Thus I identified the major decisions for each PS and the argumentation for these decisions: what is considered by the student to be a problematic situation; the strategy chosen to solve the problematic situation, sometimes with supportive and/or predictive argument; the strategy implementation, sometimes with verifying argument; and finally, the conclusion which could be the assessed and given a valuation of.

4. Belief Indication (BI) As a tool for the analysis of dynamics of mathematical thinking , Belief Indication (BI) was introduced. Belief indication is data carrying information about the person's belief as defined earlier. Data containing traces of the student's argument were marked. They could be local (for instance, a specific strategy choice) or global (e.g. belief about problem solving). BI could be explicit metacognitive statements in the video recording, interview 1, interview 2 or the questionnaire. They could have an emotional element connected to them such

[1]PS, SC, SI, C

as a gesture (e.g. a sigh) or the mentioning of a feeling (e.g. “I don’t like this’). Explicit BI can be whole or part of several beliefs.

BI can also be implicit and not always as straightforward as the explicit metacognitive statements. Beliefs could be hidden in the students’ behaviour. For example, if someone suddenly decides that an ongoing solution (correct or not) is wrong without any clear reasons, it could happen as a consequence of an underlying belief saying that an answer should be produced within a short amount of time otherwise the strategy is incorrect. “If ten minutes has passed and I don’t have an answer, I must have chosen the wrong algorithm”. The data where beliefs were not directly explained were marked. I have left out the passages where it was not clear at all, or several possible beliefs simply because of the risk of speculations. Any BI that was marked but not entirely clear was compared to beliefs from previous research. The items in the questionnaire were collected in the main groups and summarized. Then I triangulate looking for confirming and disconfirming evidence by looking at external e.g. gestures) and/ or internal (interview) evidence. A structure was formed by using BI as a coding scheme and the four steps of reasoning as a representational scheme. In combination, the different type of data collection could give a clearer picture of the students’ beliefs.

5. Reasoning The reasoning in the problematic situations was characterised using the framework described in earlier or classified as a new type of reasoning. The focus for the characterization was on the central argument for the decisions being made.

6. Connecting beliefs and reasoning Finally, the connection between the beliefs and the reasoning concerning the central decisions was made.

The analysis will be presented in these six steps for each problematic situation. Among 20 solution attempts observed, ten problematic situations constitute the data used in this paper. They were chosen to be representative for all the four students, illustrating their work. All tasks are found in Appendix A.

4. Analysis

4.1. Sam. For Sam, task one and two will be analysed. In the quotes, I stands for the interviewer and S for Sam.

4.1.1. *Sam task 1.* Sam is trying to solve following task: Find the largest and smallest values of the function $y = 7 + 3x - x^2$ on the interval $[-1, 5]$. His work is divided in four parts:

(1) Sam starts by trying to use the formula for a quadratic equation.
(2) Sam then chose to calculate the function values for the end points of the interval.
(3) He continues by looking at the graph on the calculator and investigates it with some of the tools of the calculator.
(4) He finishes by exploring some more tools of the calculator.

Part 1. *Sam starts by trying to use the formula for a quadratic equation.*

Sam reads and writes down the task. He is silent for 20 seconds. On the question if he got any idea how to solve it, *he says that he is a bit doubtful*, but continues "I will do some calculation and see what I get". He writes $-x^2 + 3x + 7 = 0$, hesitates for a few seconds and make a cross over $= 0$.

I: What is the thinking [behind this]?
S: A quadratic function.
I: You put it equal zero and then crossed over it ....
S: Yeah, I just made a correction.

Sam writes $-3/2+$, crosses over $+$ and writes $-7 = 0$. He is then quiet.

PS1: How to solve this task?

SC1: Search in the memory for some procedure. Solve it with the formula for a quadratic equation.

SI1: Can't remember the formula completely.

C1: Give up. New method.

central decisions: The central decisions in this problematic situation are the strategy choice (Sam's first in order to solve this task), and the conclusion to give up this strategy.

BI: There is information about the strategy choice in the observational data. He starts by saying he is a bit doubtful, a hint of a belief dealing with a safety issue. In the first interview, the issue of safety was confirmed when he says that the algorithm was chosen "*... because it is... safer*". He continues explaining:

S: *I have the values you need for a quadratic equation. Then I test... just to be sure.*

In the second interview, he says:

S: Here I... couldn't remember any other way to solve the task. And *then I prefer to use something I'm familiar with and know something about.*

Together, these BI:s indicates following belief: a well- known algorithm is safer.

reasoning: There is no signs Sam considers the intrinsic mathematical properties when he choose his strategy. The main argument for Sam's choice of strategy is found in the observation: "I will do some calculation and see what I get". Sam tries to recall a certain algorithm, chosen not because he know for sure it will solve the problem but might produce possible answers. There is no sign of any planning in beforehand, or an analysis of the task. Since the Sam jumps quickly from PS1 to PS2, there is no evidence that he performs some evaluation or control of his strategy choice or conclusion. This situation is classified as Delimiting Algorithmic Reasoning (DAR).

beliefs and central decisions in reasoning: Sam expresses a belief that he prefers to use something that feels safe, something familiar. His decision is a well-known algorithm chosen in order to explore. Implicitly, this behaviour gives us information about what he is not doing: reasoning. Since his behaviour doesn't indicate understanding, planning, evaluating nor controlling the task, it is likely that Sam is not even trying to perform some mathematical reasoning. Instead, he is following an algorithm chosen because he believes it is safe. His conclusion to give up, instead of exploring or evaluating this algorithm, is consistent with this since reasoning is not included in this belief.

Part 2. *Sam then chose to calculate the function values for the end points of the interval.*

After a few seconds of silence, he picks up the calculator and calculates y(5). He makes an error by calculating $(-5)^2$ instead of -5^2. *He writes down the answer 47 and is silent for a few more seconds. He adds "max = 47".* Sam writes $x = -1$ and *"min="*. He is still holding the calculator as if he considers using it, but decides to calculate $y(-1)$ in his head. He arrives with the number 5.

PS2: Is there another way to get the smallest and largest values of the polynomial?

SC2: Search in the memory for some procedure. Two values will be obtained when calculating the function values for the end points of the interval.

SI2a: Use the calculator to calculate $y(5)$. Make a vital error.

C2a: Evaluate 47 as maximum value. Assume $x = -1$ gives minimum value.

SI2b: Calculate $y(-1)$ in the head.

C2b: Must be minimum, and it is smaller than $y(5)$. New method.

central decisions: The central decisions are the conclusions (assumptions about a numerical value and despite arriving with two answers, to try another algorithm).

BI: There is information about beliefs in the conclusions. The observation shows that Sam has an expectation on the answers he is obtaining. After he got $y(5) = 47$, he assumes that this must be the maximum value without knowing the function value for the $x = -1$. Sam then states, without doing any further calculations, that $y(-1)$ must be the minimum value. These conclusions are signs of two more general beliefs: answers to mathematical task are often of a certain kind (e.g. a maximum value of a function is a numerical value between 10 and 50), and conclusions like this can be drawn in mathematics (e.g. you can from a numerical value judge whether or not it is a maximum or a minimum value of a function). Both these beliefs are expectations; what is a plausible answer to a specific task and what is a plausible way to behave when doing mathematics.

reasoning: In the second interview, Sam says that his reason to change strategy was "That was mainly a guess. An attempt to get something." and the use of the end points of the interval because "It was the only thing left I could use. There is no signs of planning in before hand, no analysis of his work or control of the answer obtained. This was just "a guess". Also, there is nothing pointing at some evaluation of the decisions. If an evaluation was made, it would have been fairly easy to discover that the answers were wrong. This situation is classified as Delimiting AR (DAR).

beliefs and central decisions in reasoning: The beliefs indicated are: (1) answers to mathematical tasks are often of a certain kind; and (2) in mathematics, conclusions can be drawn from a numerical value. They underline an Algorithmic Reasoning (AR) where you don't pay attention to the intrinsic mathematical properties of the task such as what $y(5)$ and $y(-1)$ stands for. Instead, the focus is on the numerical value of 47 influencing the central decisions.

Part 3. *He continues by looking at the graph on the calculator and investigates it with some of the in- built functions.*

(Six minutes passed)

Sam looks at the graph at the calculator. He rearranges the settings for the window.

I: What does the graph tell you?
S: It shows me something here. . . going to see a bit. . .

He goes to 'math' to get the built- in minimum- function, but when he press 'min' he gets an error message. Sam goes back to the graph, and he studies it for forty seconds with 'trace'. He can read on the window that the maximum value is about 9,25 when x is 1,5. He goes to 'calc' and the minimum- function when the interviewer asks him (seven minutes passed):

I: What does the graph tell you?
S: Now I think. . . yes, I can get a maximum value.

Meanwhile he is looking at the graph. He leaves the minimum function and goes back to the graph and study it by pressing the arrows to the right and to the left.

PS3: Is it possible to get the answers from the calculator?
SC3: Use some of the task related tools.
SI3: Able to get the maximum value. Don't remember how to use the minimum function tool.
C3: No answers. Give up. Some other tools?

central decisions: The central decisions are the strategy choice and the conclusion.

BI: Sam decides for a third attempt even though he established two answers. In the second interview he explains why he decided to look at the graph:

S: *It is much easier if you have a graph to base your assumptions on; to get it on the calculator and see.*

In the first interview, Sam claims that this would have been his preferred strategy choice but the reason why he is not successful is due to him not being able to remember exactly what to do, saying that *. . . it is a matter of remembering the stuff.* This is confirmed in the questionnaire where he partly agrees with the statements

‘Mathematics is facts and procedures you have to learn by heart.’, and ‘The key to success is to remember what you are suppose to do.’. He also agrees to ‘Mathematics requires a mental effort’, and it is more likely the effort is about memory than reasoning.

Sam expresses indications of following beliefs: (1) visual representations are useful in mathematics; and (2) memory is the key to success.

reasoning: In the first interview, Sam talks about his main thought behind this strategy choice. He says his plan was to “ Get the graph and then get a maximum value.” and “...and max and min you can get through the calculator.” Sam tries to recall a specific algorithm, how to press the buttons. His own explanation to why he wasn’t successful is because he “Kind of repressed it [how to do it] during school break” and while he is saying this, pointing at the graph and the maximum value. He confirms that his strategy choice by saying “This is actually how I would have done it.”. He is now referring to a familiar strategy; this is his preferred choice, but he can’t remember the algorithm.

There is no sign that Sam does anything with the information he obtained, neither control nor elaborating of this piece of mathematical substance, even though he studies the graph a relatively a long time (1 minute 47 seconds). There is time for reflection but there is a lack of meta- cognition. Sam renounces his solution even though it would have been possible to make some progress (e.g. compare the maximum value he explicitly notice with the answers established earlier). His conclusion is to end this attempt and try to find some other algorithmic tool on the calculator. This situation is classified as Familiar Algorithmic Reasoning (FAR).

beliefs and central decisions in reasoning: Sam expresses a belief that visual representation is useful when doing mathematics, but he doesn’t do anything with it. His main decision turns out to be the conclusion. Instead of elaborate with the information he clearly gets from studying the graph, he chose to continue his search for an algorithm. His conclusion, to abandon the algorithm and choose a new one, can be explained by his belief that memory has an impact on whether you are successful or not since reasoning is not included in this belief. Either you remember algorithms, and then you can test them, or you don’t. There is no way to work yourself to a solution by using intrinsic mathematical properties.

The strategy choice and the argumenation is consistent with Familiar Algorithmic Reasoning (FAR), but the conclusion leads him to Delimiting Algorithmic Reasoning (DAR).

Part 4. *He finishes by exploring some more tools of the calculator.*

Sam starts to press different buttons. He looks at 14 different algorithmic tools in 40 seconds before he gives up:

S: No, I don't know.

[...]

S: I could answer with this [max = 47, min = 5]
I: Is this what you would have been answering on a test?
S: If I have been working as I have been doing now, yes.

PS4: Is there some other functions that could solve this task?

SC4: Go through quickly 14 different tools.

SI4: Not successful.

C4: Give up. Choose the answers obtained earlier.

central decisions: The central decisions are the strategy choice and conclusion.

BI: It is most likely Sam is searching for an algorithmic tool. He looks very quickly (40 seconds), and there is no time for reflection or evaluation of what each tool can provide. Some of them are not even relevant to the task, e.g. 'stat plot'. In the second interview Sam says following about algorithms and trying to produce an answer on a test:

S: *You have certain methods that you know works. You don't want to hand in a blank sheet. I have hardly ever done that.*

In the questionnaire there is two statements that can be related to this problematic situation. Sam fully agrees to the statement that there is several ways of solving a task. He also says that he can't work a long time (more than 15 minutes) in order to produce an answer to a problem. Here, he gives up after 8 minutes 51 seconds.

The beliefs indicated are: (1) an algorithmical view of mathematics as a subject; and (2) there is a limit in terms of time spent on a mathematical task.

reasoning: Sam goes through 14 tools that are not related to the task. He is fast, but not successful. There seems to be no connection to the task at all in this strategy choice, and the conclusion also appears to be disconnected. He chooses the answers obtained earlier saying “I could answer with this”, showing no certainty that this is the correct answer. There are no signs of planning or analysing the task, and there is no evidence that Sam performs some control of his solution attempt. This situation is classified as Delimiting Algorithmic Reasoning (DAR).

beliefs and central decisions in reasoning: Sam strives to find an algorithm, and he repeat this until giving up. This repeated behaviour goes hand in hand with an algorithmic view of mathematic. There is nothing about reasoning here: if you believe there is several ways of solving a task and you know you have certain methods that works, then you try them (until you reach your time limit) because you don't want to hand in a blank sheet.

4.1.2. *Sam task 2.* The task is the following: Determine the equation for the tangent to $y = x^2 + x - 1$ at the point $(1, 1)$. Sam's solution is considered as one problematic situation: *Sam solves the task by using a tool of the calculator.*

Sam reads the task and is silent for 20 seconds. He picks up the calculator. When the interviewer asks him about his plan he says that he is going to use the tangent function tool on the calculator.

S: $y = 3x - 2\ldots$ would be my answer.
I: Umm. [silence]
I: Ehhh... if it was a test, would this be your strategy?
S: I would probably do this.

PS: What algorithm can be applied to this polynomial?
SC: Search in the memory for some procedure. Use a tool of the calculator.
SI: Straightforward.
C: Read the equation for the tangent on the window, $y = 3x - 2$.

central decisions: The central decision is the strategy choice.
BI: In the first interview, Sam is asked about this strategy choice. He says it is a common strategy, but “It depends on the test” whether if it is accepted by the teacher. He explains:

S: *The last questions, that would be zero points. Maybe a half [point].*[2]
I: But it pays off for you to do this?
S: *Yeah, I would think so.* Can develop on paper pretty good as well.

"To develop" means to "Sketch the graph and the tangent and so." There is an extrinsic motivation to choose this strategy, a pay off in terms of scoring on a test. Sam gives also an indication of an intrinsic motivational belief: he is not capable to do anything else. He explains in the second interview:

S: *I knew that... I haven't been that good in writing tangents and stuff like that* but here I got it quite quickly that the *only way for me to solve it was by drawing the tangent* [on the calculator]. But that I remembered. But if it is correct is another story.

The reason why he decided to choose this specific method was "the tangent you can get on the calculator and the equation [to the tangent] is in the corner and then *it feels like it [the calculator] solves it [the task] for you pretty fast*, and this was one of the primary reasons *"... because I'm not so sure myself how to calculate one of these on paper. [I] Feel then that you can get it on the calculator. You get the answer very easy* and you know pretty much how it [the calculator] works. [I] Feel then that... this is the first thing you do [in order] to get the answer."

Sam is asked if this behaviour is an instinctive action:

S: Yes. Well.. *it is what you do in class. As soon you see a task [like this] you do the connection and you use it [the calculator] immediately. Become a robot*. [laughter]

The beliefs indicated are: (1) the calculator is a safe option; (2) the calculator is a fast and efficient (in terms of producing an answer) tool ; and (3) A non- imitative method, e.g. creative mathematical reasoning, is not an option.

reasoning: The strategy choice is to search in memory for some procedure and a certain algorithm is recalled, in this case a tool on the

[2] In Sweden it is quite common with exams that have easier questions in the beginning of the test (worth 1 point) whereas the last ones are harder (3 points) but also requires a more detailed presentation of the solving procedure. You have to get a certain amount of points on the exam to pass.

graphic calculator, knowing that this algorithm will solve the task. The implementation is straightforward. Sam explains his reasoning by saying that he "got it quite quickly that the only way for me to solve it was by drawing the tangent [on the calculator]" and his strategy choice was successful because "But that I remembered". The task is identified as being familiar and the strategy choice is given by this identity. There are no signs of any consideration, not even an attempt, of the intrinsic mathematical properties of the task. He says: "if it is correct is another story". This situation is classified as Familiar Algorithmic Reasoning (FAR).

beliefs and reasoning: Sam's experience says that he can't solve it in any other way than using a tool on the calculator, which makes this a safe option. He is not even considering another way to solve it. The active goal is to solve the task and the pay- off is a correct answer in no time with little effort since you get the answer, according to Sam, very easy. This answer can result in points on a test. There are no mathematical grounds in terms of well-grounded arguments for his decision. This is what you do: you "become a robot" by using a Familiar Algorithmic Reasoning (FAR), in this case an algorithmic tool on the calculator.

4.1.3. *Sam questionnaire.* The response is not unequivocal, except for the two statements in the category "The focus in mathematics is to get the correct answer". Sam is negative to both of these statements, and this could be formulated in the belief that "To arrive with the correct answer is not the focus in mathematics." Sam's answers are partly self-contradicting, something that can be illustrated by his response in the second interview where he explains his beliefs about himself as a student doing mathematics:

> S: *I have always been a person that comes up with my own solutions.* Done completely different sometimes. *Not always calculated in the same way as the others, but instead finding short cuts and so.... Sometimes it [the solutions] is completely wrong and sometimes [I] get it [the answer] right a little bit faster.*

Sam sees himself as a creative person in mathematics, but this is probably not about mathematical reasoning. In this study, he shows no indications of creativity about working with mathematical substance to draw conclusions. It seems that creativity is more about trying, without consideration to the intrinsic mathematical properties of the task, different algorithms.

4.1.4. *Sam summary. Task 1.* The first strategy choice was chosen influenced by a belief based in security. The algorithm felt safe to Sam in a situation where he wasn't sure what to do. When not successful, Sam doesn't perform any investigation of the algorithm nor control. Instead, in the second part, the algorithmic reasoning was emphasized even further by the conclusions based on expectations about mathematical tasks. Although establishing two answers, Sam continues his search for an algorithm.

Sam's third strategy choice is explained by his belief that a visual representation can be helpful when solving mathematical tasks. Even though he expresses this belief, he does not do anything with the graph or the information obtained. He gives an impression of having a higher faith in the tools of the calculator, even higher than his own capacity to reason in mathematics. Sam says in the questionnaire that mathematics requires a mental effort, but the analysis of his behaviour indicates that he does not mean reasoning. It seems more likely that Sam thinks that mathematics is about remembering what algorithms solve which task, and which tool of the calculator that can be used, or as he describes it himself, "being a robot". His conclusion is to look for some other tools on the calculator. At this stage, it would have been fairly easy for Sam to do something with the information that he got from looking at the graph, such as to check the answers obtained earlier or to do an estimation.

His final attempt to find an algorithm, supported by interview responses, shows an algorithmic view of mathematics, that fits well with Delimiting Algorithmic Reasoning (DAR); he picks one algorithm, tries it and if it is not working, leaving it without any control or evaluation and picks another one. This general strategy prevents him from producing a more creative mathematical reasoning.

Task 2. Sam's reason for choosing this method was based on his experiences. He knew that he wouldn't be successful using an algebraic algorithm. Therefore, he decided to use a tool on the calculator that he knew would solve it in no time. Sam gets a pay- off for his strategy choice expressed by intrinsic motivational beliefs such as "this is the only way" and extrinsic since the out-come would be potential points on a test. As

Sam says in the second interview, he wants to present something (part or a whole solution, and/ or answers). Whether the strategy is correct or not from a mathematical point of view seems to be of subordinate importance. Sam's reasoning is classified as Familiar Algorithmic Reasoning (FAR), and his own explanation is memory. He remembered how to solve it, meaning which buttons and in which order they should be pressed.

Sam shows no emotions and is calm throughout the observation. The major difference between Sam's solution attempts of task 1 and task 2 is whether or not he remembers which algorithm to use. When memory fails, he turns to a count-down of different algorithms, going from Familiar Algorithmic Reasoning (FAR) to Delimiting Algorithmic Reasoning (DAR).

4.2. Paul. Task one will be analysed. Paul made two attempts to solve this task; the second time he was asked by the interviewer. In the quotes, I stands for interviewer and P for Paul.

4.2.1. *Paul task 1.* Paul is trying to solve the first task: Find the largest and smallest values of the function $y = 7 + 3x - x^2$ on the interval $[-1, 5]$. His work is presented in two parts:

(1) Paul solves the task by using the calculator.
(2) Paul solves the task using differentiation.

Part 1. *Paul solves the task by using the functions of the calculator.*

Paul reads the task. He says:

> P: Such a long time ago. [Christmas] Break and stuff. Have to think.

[...]

> I: Do you have an idea [what to do]?
> P: Know the zero points in this interval. To be able to know 'max' and 'min'.
> I: Zero points, what are...?
> P: Where the derivative is zero. Kind of.

[silence]

Paul uses the calculator to look at the graph for the function $y = 7 + 3x - x^2$.

I: [Do you have] A plan?
P: Not at the moment. Writing [plotting it] so I can see what it looks like.

Paul changes the setting on his calculator. He looks at the graph, and explores it with the help of the arrow tangents. He says "Minus one to five.", and changes the settings of the window to fit the interval, so that x goes from -1 to 5.

P: Then it is just to check what is the highest [largest] and the smallest value here. The derivative is zero and it is just to check on the calculator for instance 'calc' and then 'maximum', 'left bound'. Maximum is [when] x [is] $1,5$ and then y [is] $9,25$.

While he is talking he points at the graph on the calculator. Paul asks if he has to write it ("according to the calculator") down on the paper. The answer is that he doesn't have to. The interviewer asks him if he is satisfied with his solution, and Paul replies he is.

I: [What about the] Smallest value?
P: Has to be to the far right because there is not even room for it [the graph on the calculator].
I: Are you sure?
P: That's how it should be!
I: Yes. . . ?
P: [It's] Obvious. Since it is the smallest value of y you can get in this interval.

PS1: How to solve it?

SC1: Search in the memory for some procedure. Solve this task using the calculator: using a tool on the calculator and by studying the graph.

SI1a: Maximum value by using a tool on the calculator. Straight forward.

C1a: $y(1,5) = 9,25$

S1b: Minimum value by studying the graph.

C1b: The correct end- point of the interval.

central decisions: The central decision is the strategy choice; his first decision in order to solve this task.

BI: In the second interview, Paul is asked about his decision:

> P: *I know what I am suppose to do, but I don't know how to do it.* It is more like that.
> I: What made you choose the calculator?
> P: *It is easiest.*
> I: A common choice?
> P: No. *Normally I try to solve it algebraically because... well, it gives more. It is more maths.*
> I: Gives more points [on an exam]?
> P: No more like, *it gives more... you learn more from it. Otherwise you can sit and calculate all the tasks with the calculator and then you don't learn much about why you do certain things.*
> I: So, with the calculator you don't use your brain as much?
> P: Yeah, I think so. *Especially with graphs.*

Paul gives an intrinsic motivational belief as a reason for choosing this strategy 'I can't reason myself to a solution', and his external motivation is to solve the task. In the second interview, Paul explains further why he used the calculator to differentiate a function. He says that he has "*Difficulties with differentiation [the definition]. Failed on a few exams. I'm doubting. My own knowledge.*" and that the calculator "*Feels like a huge help, it does actually*". Paul explains why he prefers solving tasks algebraically than using the calculator. He says that he rather use "the methods that you have used mostly in your life". He continues:

> P: It is more in the first grade [of upper secondary school] when you get the calculator, and *then it feels like you solve almost everything with the calculator, and I think that you loose a lot of this... the actually learning bit of different methods*, I think. Since they [the algorithms] are already in the calculator, but *there is a lot of tasks that would take flipping long if you didn't use the calculator.*
> I: You mean for instance, drawing [graphs] by hand?
> P: Yeah, or if you had a lot of decimals in a task and stuff like that. It's not so flipping difficult, but it is just the thought of... well, *a lot of decimals is just hard [work].*

The beliefs indicated are: (1) I can't reason myself to a solution; (2) you don't learn as much mathematics if you use the calculator; (3) the calculator saves time and work for you; and (4) the calculator is a safe option.

reasoning: Paul knows what he is suppose to do, but not how to do it. His strategy choice is to search in the memory for some other procedure, using the calculator as a starting point and performs two different types of reasoning. This is a familiar task and Paul identifies a certain algorithm to solve this problematic situation. Once the strategy is decided, the implementation is straightforward. Pauls appears to be confident and arrives with two correct answers (the largest value when $x = 1,5$ and the smallest when $x = 5$). This situation is classified as Familiar Algorithmic Reasoning (FAR) and there is a local Creative Reasoning (CR) when Paul is studying the graph to get the minimum value of the function. His argument is based on the intrinsic mathematical properties: "it is the smallest value of y you can get in this interval".

beliefs and central decisions in reasoning: Despite Paul's beliefs that you don't learn as much mathematics when using the calculator, the pay-off is greater in terms of it is a safe and easy option, and it saves time and work for you especially when you are not sure how to solve it algebraically. Paul's choice to use the minimum function tool grounds in these beliefs, which results in Familiar Algorithmic Reasoning (FAR). But he is also able to produce a local Creative Reasoning (CR) by studying the graph.

Part 2. *Paul solves the task using differentiation.*
Paul then worked with task 2 and 3. In the first interview, the interviewer asks Paul to solve the task one more time. Paul starts by saying that the differentiation should be equal to zero. He differentiate the function, arriving with $3 - 2x$. The interviewer asks him what he is going to do with that:

> P: Well, it shouldn't be correct since *it should be like two roots.*
> I: Yes?
> P: Well, I mean, when I say that the derivative is zero, then I mean that the slope is zero.
> I: Yes?
> P: Mmmm. But that should be... alright, *that could be two*

anyway. Because it could be... ehh... since it is minus, it is $3 - 2x$. Wait a minute! You should be able to calculate $3 - 2x$ when that is zero?
I: $3 - 2x = 0$. Yes, it is possible to calculate that.
P: *That one should have two solutions.* I think.
I: You think. Yes?
P: *One positive and one negative even.* That's what I think.
I: One positive and one negative. Why a positive and a...?
P: *Because it should be a maximum and a minimum.* But then also, it is a minus sign infront of it [x^2], but it may as well be a plus sign. If you write it here...

Paul writes: $3 - 2x = 0$

P: *But wait a minute.*

[silence 40 seconds]

P: No.
I: Why do you say no?
P: *When I think that this should be zero, then it is just one way and that is 1,5.* After that there is not much that makes it zero. Then it is just going to be more or less. *If you take it [1,5] negative, it [the function] becomes another number.*
I: So $x = 1,5$, that is completely wrong?
P: Yes.

PS2: How to reach two solutions out of $3 - 2x = 0$?

SC2: Tries to relate to local steps of an algorithm.

SI2: Hesitating.

C2: Expects two x- values, and the result is one. Therefore, this must be wrong.

central decisions: The central decisions are the implementation of the strategy and the conclusion. When Paul is implementing the strategy, he makes an evaluation from which he draws a definite conclusion.

BI: Paul first seems to know what he is talking about from a mathematical point of view. But the more he develops his solution, the more he reveals there is more than just the intrinsic mathematical properties that grounds his reasoning: his expectations. He says, while he is solving the task, the equation should have "two solutions" and they should be "One positive and one negative even", because "it should be a maximum and a minimum". In the second interview, Paul confirms these expectations:

> P: *I thought it would be two turns [on the graph]... last thing we worked with [in class].*

Here, Paul refers to a specific situation, a third degree function, which they worked with in class. In the observation, the conflict between Paul's expectations and the algorithm becomes visible. His hesitation (40 seconds) shows there is a problem. In the second interview, Paul tries to explain:

> P: Why I hesitate... I don't know why I hesitated. *I think it is because I expected two answers. A "square root thing"*. But that is wrong.

Here, Paul emphasizes a specific local step of an algorithm. His hesitation is the result from the discrepancy between the intrinsic mathematical properties and what he tries to recall about this local step. The beliefs indicated are: (1) an important base for strategy choices is to recall local stages of an algorithm; and (2) mathematical tasks are often similar.

reasoning: The problematic situation is given by the interviewer. Since the task is familiar, and the strategy should solve the task, Pauls hesitation is strong. His conclusion is drawn from expectations: since I just got one x- value, and I expected to get two, this must be wrong. There is no well- grounded mathematical arguments for this decision. This situation is classified as Familiar Algorithmic Reasoning (FAR).

beliefs and central decisions in reasoning: Paul expresses beliefs about what he expects the algorithm to look like. His hesitation, when evaluating the algorithm, indicates there is a major conflict between the algorithm and these beliefs. The conclusion then is definite: since this is not what I expected it to be, it must be wrong.

4.2.2. *Paul questionnaire.* The conclusion of the questionnaire is following three categories:

- "The focus in mathematics is to get the correct answer" where Paul is negative to both of these statements, and this could be formulated in the belief that "To arrive with the correct answer is not the focus in mathematics."

- "Mathematics is a mental effort that takes time" which consists of five statements. Paul is positive to four of them.
- "To do well in mathematics requires a good memory" consists of four statements. Paul is positive to three of them.

Paul's response to the questionnaire gives an ambiguous picture. On one hand, memory is a major strategy but at the same time understanding is one too. Paul explains in the second interview:

> I: It [mathematics] is a lot to memorize?
> P: *To remember why.*
> I: Why?
> P: *Why you do certain things.*
> I: Do you spend a lot of time [doing this]?
> P: Yes, I feel *it is more important then to finish the weekly assignment.*
> I: Weekly assignment?
> P: Yes, we have that sometimes. *Sometimes it just feels pressing and stressing since if you are supposed to sit down and do a lot of tasks, and you don't understand completely, then it doesn't give that much.*[...]*It has to take its time.*

But when talking about the statement 'I learn mathematics by solving a lot of tasks that are similar to each other', which he agrees almost fully to, Paul stresses how important it is to remember:

> P: Well, *that is when you have understood, then it is just to rub it in*, or how should you say it, *so it comes to me by heart so when you see it [a similar task] saying "You should do this", then you know "Yeah, this is how I should do it".*

According to this, understanding is not enough. You still need to "rub it in" so it comes to you "by heart" not leaving much space for creative mathematical reasoning.

4.2.3. *Paul summary.* Although Paul claims there is a major draw back using the calculator, the pay- off is greater. It is a safe and easy option when not knowing what to do, and it saves time and work. There is also indications of an intrinsic motivational belief saying that reasoning is not an option for Paul in this situation, despite the fact that he gives examples of intrinsic mathematical properties of the task. In the second part, Paul talks a lot about his expectations of the algorithm. He gives a good picture of what he believes the answers should look like. Even if he has seen the graph and worked with it to draw conclusions and to arrive with the correct answers, he still expects it to have a specific look ("two turns", a third degree function), because that was the last thing they worked with in class. It is also interesting to note that Paul explains (more or less correctly) what it means when a derivative is zero. This single piece of knowledge combined with the information about how the graph look like, would have been a good starting point for a creative reasoning. Paul has solved the task and arrived with the correct answers, but his expectations of what the answers should look like overtakes his mathematical reasoning. When the algorithm doesn't lead to the expected answers, Paul concludes it must be wrong.

4.3. Ella. In the quotes, E stands for Ella and I for the interviewer.

4.3.1. *Ella task 1.* Ella worked with task one: Find the largest and smallest values of the function $y = 7 + 3x - x^2$ on the interval $[-1, 5]$. Her work consists of three problematic situation::

(1) Ella tries differentiation.
(2) Ella talks about x- values.
(3) Guidance by the interviewer.

Part 1. *Ella tries differentiation.*
Before the camera started recording, Ella says something about that *she is nervous and doesn't like to make errors.*

> I: Why don't you like to make errors?
> E: *It is hard.*
> I: It is hard. On/ For you?
> E: Yes. *I think I should be able to do it [correct]. I had good grades* on both the A- level and the B- level. I had PSD [3] on both A and B, *but now everything feels so hard.*

[...]
Ella makes the decision to proceed with the problem solving. She says "Ok, I'll do it." and reads the task.

> I have to write it like this, have to do it.

[3] pass with special distinction, the highest grade

Ella writes $y = 7 + 3x - x^2$. [...] She continues writing $[-1, 5]$. Then she says:

> E: *Shall I differentiate or what? I don't understand what I should do. It's probably that.*

[silence]

> I: What is your first thought?
> E: I'm thinking that I should... yeah, differentiate it first.
> I: Because...? Why do you think that?
> E: Well... [I] *know that we normally do that. That is the first thing you do when you have a task where you should put values* [in a formula] I assume.
> I: How come you think about differentiation? What make you think...?
> E: *That is because it says x and like x^2. Then it is good to differentiate.*

[talk about the interval]
Ella writes: $y' = 3 - 2x$

> E: That is what you get if you differentiate. You take away the number and then... Then I'm not sure if I should put in that [the interval] in the first formula or in the differentiation formula. Because that I can't remember. But I can try.

PS1: How to solve it?
SC1: Search in the memory for some procedure. Differentiation.
SI1: No problem. Arrive with $y' = 3 - 2x$
C1: Unsure what to do with this.

central decision: The central decision is the strategy choice.
BI: There are two sides of Ella's strategy choice. One part is what Ella believes what you are expected to do when "it says x and like x^2". This is what they "normally do". There is also information about her emotional status. Ella has negative emotions connected to her performance, and there is indication of a belief about herself as a student: "I think I should be able to do it [correctly]. Ella's own argument for believing this is her previous grades. This expectation is connected to the second part of the strategy choice,

which is what is safe to do. In the observation, Ella turns to the interviewer and asks if a specific method is correct, an indication of safety/ security. In the second interview, she explains why she wants to follow an algorithm, especially on a test:

> E: *I like to do things that I know is working. When I'm doing some exams and so. Otherwise it doesn't matter. Then you can try anything you like.*
> I: Oh, ok. So if you do an exam...?
> E: *Then I want to, because I know that this is what they accept. The right method, and then I know I'll get some points.*

Ella confirms that when solving a specific task, there is a specific algorithm that she prefers to use. This is motivated by an external cause: get some points on a test. There is an indication of an intrinsic motivational belief saying that she doesn't trust her own reasoning well enough to use it on an exam. There is more information about this belief in the questionnaire where Ella agrees partly (almost fully) to 'I have enough of mathematical knowledge to be able to create my own solution to a mathematical task'. She explains why this is true at home but not on a test in school:

> E: *It is easier to work at home.*
> I: But you wouldn't to this...
> E: ...on a test. [Ella fills in]
> I: You don't trust your knowledge on a test, but home?
> E: *That's because I'm scared of getting a bad result on the test. Because I don't want to have bad grades.* Maybe not a good way of thinking. *I just really want to do well, I want to have good grades so that I can get in to [study] what I want to later, if I want to continue my studies later on. Then it is more important to do in the way they [the teachers] want you to do instead of what you might dare to do.*

This is confirmed in the questionnaire where she fully agrees to the statement 'The purpose with mathematics education is that the teacher tell you which methods you should use to solve certain tasks. Later on in the second interview she comes back to this issue:

E: *I trust that I understand those methods they have taught us. It is just my own [methods] I don't completely trust. Most of the times what they say are true.*

The beliefs about safety/ security are confirmed in the second interview where she emphasizes the importance of doing the right way to make sure that she passes the exam, which is the external motivation saying that "*You can always try [out your own reasoning at home], but not on an exam, I wouldn't do that because then you can do poorly* [on the exam]". She supports this with an emotion of being scared; it is an action of security.

E: *I'm scared of that I'm going to do poorly. If you have tried a method at home, then you don't know if the teachers thinks it is weird. If it will do or not.*

She ends by motivating why she prefer to study mathematics instead of other courses saying "*I rather do maths and so. Then I know I will pass.*" Although Ella doesn't trust her own reasoning she expresses an extrinsic motivational belief why she wants to study mathematics; she can pass the course.

The beliefs indicated are: (1) I should to do well in mathematics; (2) familiar strategies are safer; (3) mathematical tasks should be solved in a specific way; (4) my own mathematical reasoning is not a safe strategy; and (5) I can pass the courses in mathematics.

reasoning: The task is identified as being familiar, and can be solved by a certain algorithm (differentiation). The strategy choice is based on surface reasons, by the appearance of the function and with no consideration of what type it is or what you get from differentiation such a function. The implementation is straight forward at the beginning, but when the algorithm does not look like as Ella expected, she gets confused. This situation is classified as Familiar AR (FAR).

beliefs and central decisions in reasoning: Ella expresses beliefs mainly concerning what is safe (to use a specific algorithm), but also about what she is expected to do when you have this type of function. These beliefs are supported by Ella having negative emotions about her expected performance, connected to a belief about herself as student. The strategy choice is an algorithm which you are supposed to use, a safe choice, and the outcome should be correct because it is a familiar task.

Part 2. *Ella talks about x- values*

> E: You can.... I think that you can just put in everything [all the values], but *that is going to take a long time.*
> I: You mean, put in -1, 0, 1...?
> E: *I think that I was taught how to do it in an easier way.* That is if you should get the maximum and the minimum values, maybe. *This becomes hard when I don't know what to do. How I should do it.* I'm thinking about different things. We just started with something new today. And that is a lot of things.

[...]

> E: Because if you calculate maximum and minimum then I don't get...or...it was nothing.
> I: If you calculate maximum and minimum then you don't get what?
> E: Then I get these two. I guess. That interval. I don't know how I should use it....
> I: How you should use it?
> E: Mmm. I don't know.

[silence]

> E: But if I try to put in [the values] in the formula. See if that works. *Ugh! I don't like this. When it is like this.*

[silence]

PS2: How to continue?

SC2: Search in the memory for some procedure. Maybe put in different values in the function.

SI2: Hesitates because it should be easier.

C2: Stop.

central decisions: The central decisions are the strategy choice and implementation.

BI: There is indications of a belief about expectations. Ella is not sure about this strategy since it is going to take a long time, and it should be "an easier way". The next local step doesn't look like she expected and she finds it hard when she does not know what she is supposed to do. Ella does not like to be in such a situation. She expresses the negative emotion saying "Urgh! I don't like this!". As in the previous problematic situation, Ella likes to do things she knows will work. Following belief is indicated: when doing mathematics, I should recall local steps of an algorithm.

reasoning: The problematic situation arrives from Ella's confusion, and she tries to recall how to proceed. She searches for the next local step of an algorithm but moves on to looking for different options when not knowing exactly what to do. She searches for a method and talks about putting different values in the function. Since Ella is sceptical to this strategy, the implementation doesn't occur. The conclusion is to stop working. This situation is classified as Delimiting Algorithmic Reasoning (DAR).

beliefs and central decisions in reasoning: Ella has high expectations on herself when it comes to remembering how to solve a mathematical task, or more specifically, recalling local steps. Not feeling sure, she starts a search for an appropriate second step but the implementation is on hold because of her uncertainty; her expectations differ from the reality. This is supported by negative feelings.

Part 3. *Guidance by the interviewer.*
Eight minutes passed, *Ella is getting more nervous. She acts nervously, waving her hand.* The interviewer makes the decision to encourage Ella to continue her solving attempt. During the guidance Ella says that "*because if you differentiate then you don't get a second degree function. You get that.* [points at $y' = 3 - 2x$]", and then "*... it feels like I'm lost.*" *Ella laughs, nervously.* She is asked if she wants to try to solve some of the other tasks:

> E: *I* want *to do it [to solve this task]. I don't want to do it [in this way], that I start with one thing, and do it half-done, and then I take the next thing and do it half-done. Because, then it is like this, that it is unfinished, and that doesn't feel good.*

PS3: How to solve it?

SC3: Search in the memory for some procedure. Back to differentiation.

SI3: Can't proceed.
C3: Give up.

central decisions: The central decision is the problematic situation.

BI: Ella appears to be emotionally effected: she is waving with he hand and she laughs nervously. The task does not look like she expected it, a third degree function, and she does not know how to proceed. Ella ends the observation explaining why she wants to finish solving the task, and hereby indicating a motivational belief:

> E: I *want* to do it [to solve this task]. I don't want to do it [in this way], that I start with one thing, and do it half-done, and then I take the next thing and do it half-done. Because, then it is like this, that it is unfinished, and that doesn't feel good.

The belief indicated is a solution attempt should be finished before starting with a new task.

reasoning: Apart from the decision to continue, all steps are made by the guidance of the interviewer. Therefore, this situation is classified as Person-guided Algorithmic Reasoning (PAR).

beliefs and central decisions in reasoning: Focusing on the central decision, to continue, Ella expresses a motivational belief saying you should finish a solution attempt before moving on to the next task. She locks her focus on this task, and support her decision with negative feelings.

4.3.2. *Ella questionnaire.* The general picture of Ella given by the questionnaire is an algorithmic view of mathematics. The conclusion of the questionnaire is following three categories:

- "The focus in mathematics is to get the correct answer" where Ella is negative to both of these statements, formulating a belief that "To arrive with the correct answer is not the focus in mathematics."
- "Mathematics is a mental effort that takes time" which consists of five statements. Ella is positive to all of them.
- "To do well in mathematics requires a good memory" consists of four statements. Ella is positive to three of them.

In the second interview, Ella was asked about some of the statements. Ella agrees partly (almost fully) to 'The key to success is to remember how to solve it.' and 'Mathematics is basically facts and procedures you have to learn by heart.' She says:

> E: *It depends from problem to problem. But with differentiation, then you have to learn the rules. You have to learn the rules of differentiation and so. These are facts.*

But Ella also gives another view of mathematics, talking about understandning. She disagrees to 'Normal students can't except to understand mathematics. They can just memorize it without knowing why.' and she says:

> E: Why shouldn't you understand? *You have to understand to be able to do some work on it.* If you do task, *there is no point to do tasks if you don't undestand what you do. I don't think that there are many [students] that do task and don't understand what they are doing. That is pretty unnescessarily.* [laugh] I think so anyway, 'cause I kind of feel that if I should do a task, *I want to know what I'm doing. Otherwise it doesn't give me anything. That is pretty unnescessarily.*

Here she contradicts her previous statement, that mathematics is about memorizing which method to use. It is higly plausible, Ella is talking about understanding as in being able to remember instead of being able of elaborating with a mathematical substance.

4.3.3. *Ella summary.* Ella is nervous. She acts nervously (in her voice and body language), and she says that she is nervous. It takes about two minutes before her voice goes down to normal conversational tone, and she agrees to do the tasks. It is obvious that Ella does not like test situations, something her teacher confirms. She is afraid of doing poorly (not being able to solve the tasks correctly) and she feels pressure to do well. Ella has high expectations on her self and her performance in school. She says "it is hard" because she should be able to do this, but now "everything feels so hard".

Ella shows a combination of different types of beliefs, supported by emotions and motivation. In the first part, there are two types of expectations (on herself and on what is should look like when solving tasks like this one) combined with beliefs on what is safe (to use a specific algorithm for a specific task). When the algorithm doesn't look like she expected, her negative emotions intensifies.

Without the feeling of security, she looks for other options in the second part. But, Ella doesn't continue a global Delimiting Algorithmic Reasoning (DAR). Maybe the negative feelings don't support her to step away from the initial strategy choice, the one made based on familiarity and security. She clearly doesn't like to be in a situation where she doesn't know exactly what to do. Still, she is motivated to continue. Her intrinsic motivation is that she wants to finish the solving attempt. At this stage her emotional state has overtaken her reasoning, and the interviewer guides her. Ella's focus on algorithm is confirmed in the questionnaire. In her behaviour, there is (in a test situation) no room for exploring or searching for new ideas. It is interesting to note that Ella has two different ways of doing mathematics. In school, the focus is on what is safe, but at home she would act a bit different, allowing herself to ponder on solution strategies.

5. Discussion

There have been a large number of studies investigating students' beliefs (e.g. Carlson (1999) and Svege (1997)). A few of them have looked how beliefs influence students reasoning (Schoenfeld, 1985; Wong et al., 2002). This study goes further investigating this influence looking at the central decisions made during mathematical task solving. There is an algorithmic view presented combined with an algorithmic reasoning. Beliefs appear to be rather dominant compared to the student's actual mathematical knowledge. Even when the student would have been able to make some progress, these type of beliefs have influenced the student to take another route. For instance, Sam explicitly noticing the maximum value, but his conclusions has nothing to do with this vital piece of information. He expresses a belief that visual representations are useful in mathematics, but it is not used for a creative reasoning. Similarly Paul, who concludes a correct answer is wrong because his expectations says otherwise even though he has solved the task earlier, using the graph to arrive with the exact same answer. There is a lack of consideration of the intrinsic mathematical properties resulting in Imitative Reasoning (IR).

What type of beliefs does this algorithmic view consists of? Three major themes stand out: expectation, safety/ security and motivation, the latter one including motivational beliefs and active goals which both can be intrinsic or external. Let us start by looking at expectations, which mainly influence the implementation of the strategy and the conclusion. Sam has preconceived beliefs considering the numerical number 47 which he might have because in school tasks, maximum values often are between 10 and 50. Paul's expectations of a local step of an algorithm overtake his reasoning, and Ella becomes influenced by her emotional response when she gets confused of the discrepancy between her expectations and the task. Expectations are not a new or strange phenomena, we all have them, but the problem arises when expectations become the dominating factor compared to mathematical knowledge, and out manoeuvres what could have been resulting in a more creative and productive mathematical reasoning.

Like previous research (Kloosterman, 2002; Op't Eyende et al., 2006), I find motivational beliefs an important subcategory of beliefs. From an intrinsic perspective they constrain students' mathematical reasoning presenting specific algorithms as 'the only way' for the students if they want to produce an answer. Creative reasoning is therefore not an option. External motivation can be short term as in solving a specific task to a long term goal of doing well in mathematics courses.

All three students base their first strategy choice on beliefs dealing with safety/ security. This would not be an unusual strategy choice even for an experienced mathematician. But, when the implementation of the first strategy is not straight forward, the next step becomes of interest. What do you do when your safety choice is not working? In this study, the students turn to Delimiting Algorithmic Reasoning (DAR) instead of investigating the intrinsic mathematical properties of the task any further or, as in Paul's case, rely on their expectations.

For all the students in this study, memory is a major problem solving strategy. This is in line with Kloosterman's (2002) results where the students saw memorization as an important aspect of learning mathematics. The difference between Global Delimiting Algorithmic Reasoning (DAR) and Familiar Algorithmic Reasoning (FAR) in Sam's case (task 1, PS 3) is whether or not he can remember the algorithm correctly. His argumentation concerning the strategy choice indicates Familiar Algorithmic Reasoning (FAR), but due to not being able to remember all the local steps of the algorithm he goes back to Delimiting Algorithmic Reasoning (DAR), which turns out to be his global strategy in this solution attempt. Either way, remember or not, it is limiting the possibility for a Creative Reasoning (CR). Ella goes even further saying that her own mathematical reasoning is not a safe option. Her focus on memory, to remember which algorithm to use to which task, becomes her so far very successful safety net. But she has now reached a stage where it is hard to remember all the local steps of specifics algorithms. There is a limit, which differs from person to person, how far you can get with this type of strategy.

The question arising from this finding is whether or not there is a difference between students' beliefs by those who use Imitative Reasoning (IR) and them who use Creative Reasoning (CR)? What is considered being safe by the students in this study, constrains them from performing a Global Creative Reasoning (CR). Since this study only provides one example of local Creative Reasoning (CR) it doesn't give any information about this and no conclusion can be drawn. Therefore, further research is required.

The calculator is presented as something representing security (as well as the text- book and the teacher). It is used in many situations as a safer option than Creative Reasoning (CR). The use is in majority concerning algorithmic tools resulting in algorithmic reasoning, and very little about analytical tools as help for investigating the intrinsic mathematical properties of the task. This feeling of security is given a second dimension if you also believe that you are limited in producing Creative Reasoning (CR) and/ or your own reasoning is not safe.

Another result from this study is that the concepts cognition, emotion and motivation are intertwined with each other, and there are important interrelations between them. We have seen this in previous research in trait aspect, e.g. Hannula (2004) talks about a positive change of beliefs and behaviour when a student sets a new performance goal. But even in a state aspect, the impact is noticeable. In most of the decisions in this study, beliefs depend on student's active goal and/ or emotional state and vice versa. As stated earlier, for Ella emotions becomes a dominant factor. These negative emotions derive, amongst other things, from her expectations on her self, and from the situation when the task does not correspond with her expectations on the algorithm. The latter means that her security/safety is taken away from her. Meyer and Turner (2002) concluded in their work that they can't neglect the emotional state of the students. I have to agree, even though emotions weren't the main focus of this study. Since the study wasn't designed with interrelations in mind some of the methods don't deal with this issue. For instance, the questionnaire only gives fragments of information about individual and isolated beliefs. A different type of questionnaire, or at least a broader one in terms of covering the three concepts, might be helpful for investigating this issue further.

5.1. Possible implications. What can we do about the algorithmic view and algorithmic behaviour presented above? Repeating the same type of tasks would form these types of expectations and strategy choices, by making the student facing repeatedly the same problematic situations. If the student is not exposed to different problems that provides a range of difficulties, it is hard to stimulate the intellectual development (Pólya, 1945). Op't Eyende et al. (2006) presents another aspect of this. They concluded that "Teaching students how to solve mathematical problems then implies that we have to teach them also how to cope effectively with feelings of frustration or sometimes anger" (s. 204). The combination of these two emphasizes a need for stepping away what is created being safe or what you can expect mathematics to be such as what answers usually look like, the look of an equation, or a strategy choice based on "we normally do that" which has little or nothing to do with the intrinsic mathematical properties. It should be a positive feeling being frustrated when solving mathematical problems (Hannula et al., 2004). Since good problem solvers spend their time differently when solving problems compared to less successful ones (Schoenfeld, 1985), stimulating planning, analysis and evaluation of a task would be one way escaping a dominating focus on algorithms and instead help to highlight questions such as 'what is this task really about?' and 'what does this mean?'.

There is therefore a need for students to work with different types of problems so they are able: (1) to strengthen their knowledge coping with negative emotions and feeling of being ok about *not* knowing exactly how to solve a task, or to be aware of ones meta- affect; (2) to explore and stimulate their creative reasoning; and (3) to try to break the didactical contract so you as a student can't rely on what is normally being done and also trying to avoid what Vinner (1997) refers to as pseudo-learning. Teaching students how to solve mathematical problems then also implies teaching a good various of different problems providing different type of problematic situations with the focus on the different intrinsic mathematical properties of each specific task. As a result of this, teaching mathematics would then require a good, solid background in mathematics.

We need to know more about how beliefs influence mathematical reasoning, and the next step could be to look at how students who are good problem solvers solve problematic situations. What type of beliefs would support a Creative Reasoning (CR) instead of Imitative Reasoning (IR) based on superficial grounds? In addressing this problem, we should consider that different areas of mathematics might produce different type of reasoning and maybe therefore bringing different type of beliefs to the surface. Would the same behaviour be recorded when working with geometry or discrete mathematics? Are some areas in mathematics dominated by a culture focusing on memorizing rules, whereas others might even invite students to be creative? Is there a gender difference in mathematical reasoning and beliefs? Further research is required to answer these questions, and for continuing the investigation of the complex relationship between cognition, motivation and emotion, and students' mathematical reasoning.

Appendix A. Mathematical tasks

The following tasks were used:

(1) Find the largest and smallest values of the function $y = 7+3x-x^2$ on the interval $[-1, 5]$.
(2) Determine the equation for the tangent line $y = x^2 + x - 1$ at the point $(1, 1)$.
(3) Is there a constant a so that $f(x) = ax^3+4x$ has a local maximum at $x = 1$?

Appendix B. Questionnaire

These are some of the statements on the questionnaire that the students could choose if they fully agreed, partly agreed (almost fully), partly disagreed (almost fully) or disagreed with.

- Mathematics requires a mental effort.
- In mathematics things are either right or wrong.
- The purpose with mathematics education is that the teacher tell you which methods you should use to solve certain tasks.
- The student's role is to get mathematics knowledge and to prove that you received this knowledge.
- Mathematical proofs are useful to me for learning mathematics.
- If you don't know how to solve within five minutes, you are wrong.
- To get the right answer is more important that the method.
- Mathematical problems should be solved in just a few steps.
- In mathematics, it is the answer that counts.
- Normal students can't except to understand mathematics. They can just memorize it without knowing why.

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