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Författare 1: Xiangli Meng
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Author: Xiangli Meng[♣]

Abstract: Quadratic assignment problems (QAPs) are commonly solved by heuristic methods, where the optimum is sought iteratively. Heuristics are known to provide good solutions but the quality of the solutions, i.e., the confidence interval of the solution is unknown. This paper uses statistical optimum estimation techniques (SOETs) to assess the quality of Genetic algorithm solutions for QAPs. We examine the functioning of different SOETs regarding biasness, coverage rate and length of interval, and then we compare the SOET lower bound with deterministic ones. The commonly used deterministic bounds are confined to only a few algorithms. We show that, the Jackknife estimators have better performance than Weibull estimators, and when the number of heuristic solutions is as large as 100, higher order JK-estimators perform better than lower order ones. Compared with the deterministic bounds, the SOET lower bound performs significantly better than most deterministic lower bounds and is comparable with the best deterministic ones.

Key words: quadratic assignment problem, genetic algorithm, Jack-knife, discrete optimization, extreme value theory

♣ PhD-student in Micro-data analysis at the School of Technology and Business Studies, Dalarna University, SE-791 88 Falun, Sweden. E-mail: xme@du.se. Phone: +46-23-778509.

1. Introduction

The combinatorial problems in operational research have been widely studied due to the significant utility in improving the efficiency of many reality problems. However, many combinatorial problems are NP-hard and enumerating all possible solutions becomes impossible when the problem size increases. Many studies have been devoted to developing efficient (meta-) heuristic algorithms to solve the problem and provide a good solution. A shortage of the heuristic solutions is that it is difficult to assess the quality of the solution, i.e., the difference between the heuristic solution and the exact optimum is unknown. One common strategy is to use algorithms providing deterministic bounds, such as Lagrangian relaxation (Fisher 2004) and Branch and bound (Land and Doig, 1960). This strategy is popular and reasonable for many problems, but it confines the choice of heuristic algorithms, and its performance (largely) depends on the choice of parameters. For many widely used algorithms such as Genetic algorithm and Simulated Annealing algorithm, the quality of their solutions remains vague relative to deterministic algorithms.

An alternative strategy for assessing the quality of heuristic solutions is to use the statistical bounds, which is also referred to as statistical optimum estimation techniques (SOETs). The idea of SOETs is that parallel heuristic processes with random starting values will result in random heuristic solutions, thereby providing a random sample close to the optimum. Statistical theories such as nonparametric theory and extreme value theory are then applied on this random sample to estimate the optimum and provide the confidence intervals. Pioneering work has been done by Derigs (1985) on travelling salesman problems (TSPs) and quadratic assignment problems (QAPs). It is shown that statistical bounds are competitive with deterministic ones for both problems and has more potential in QAP than in TSP. After Derigs (1985) there is some research devoted into developing SOETs, but many questions remain unanswered, hindering the wide application of SOETs. Giddings (2014) summarizes the current research and application situation of SOETs on operational problems. A problem class $\mathcal{J}$ is a group of problems instances I, so $I \in \mathcal{J}$. This class contains well-known combinatorial optimization problems such as TSP, Knapsack, and Scheduling. A heuristic H is a combination of computer instructions of the solution method with a given random number seed. The heuristic class $\mathcal{H}$ is the collection of possible heuristics. n is the number of replicates arising from unique random number seeds. The SOETs consists of all the combination sets for $\mathcal{J} \times \mathcal{H}^n$. For a complete investigation of the SOET performance, all the restrictive types $I \times H^n$ need to be checked.

For a specific combination set of $I \times H^n$, Carling and Meng (2014, 2015) examine the application of SOETs on p-median problems. They compare the performance of SOETs systematically regarding different heuristics H and number of replicates n and give the following conclusions:

(1) The SOETs are quite informative given that the heuristic solutions derived are close enough to the optimum. A statistic named SR (standard deviation ratio) is proposed for evaluating whether this condition is satisfied. The statistical bounds will cover the optimum almost certainly if SR is smaller than the threshold 4.

(2) Comparing the performances of different SOET estimators, the 2nd order Jackknife estimator and the Weibull estimator have better performance in having smaller bias and providing statistical bounds covering the optimum. When SR<4, the bounds cover the optimum almost certainly. The Gumbel estimator and the 1st order Jackknife estimator perform worse.

(3) Small sample size n, e.g., $n = 3$ leads to unstable intervals, but 10 heuristic solutions provide almost equally good statistical bounds with 100 heuristic solutions. Thus the effect of having more than 10 heuristic processes would have small effect on the functioning of SOET.

(4) Different heuristics do not affect the performance of statistical intervals. The solutions derived by Simulated Annealing are not significantly different from those derived by Vertex Substitution. The performance of point estimators and statistical bounds are almost the same as long as SR<4.

(5) Under the same computing time, statistical intervals give better results than deterministic intervals derived by Lagrangian relaxation. The statistical intervals have much shorter lengths in most of the cases while almost certainly covering the optimum.

Conclusion (1), (2) and (4) are novel conclusions, i.e., they could not be traced back to similar research results while conclusion (3) is analogous with Brandeau and Chiu (1993) which states $n = 10$ would obtain as good solutions as $n = 2000$ and statistical bounds yield better lower bounds than the available analytical bounds. (5) coincides with Brandeau and Chiu (1993) and Derigs (1985). These conclusions provide us with an effective way of deriving useful statistical intervals. However, Carling and Meng have only conducted the analysis on *p*-median problems, and left the validity of SOETs unverified on many other operational problems in $\mathcal{J}$. The focus of this paper is therefore to analyse the performance of different SOETs on another important combinatorial problem, namely the quadratic assignment problem.

The quadratic assignment problem (QAP) is a classical combinational problem in operational research. It is formulated as follows. Consider two N-dimension square matrixes $A = (a_{ij})_N$ and $B = (b_{ij})_N$, find a permutation $(x_1, x_2, \dots, x_N)$ of integers from 1 to N that minimises the objective function:

$$g = \sum\nolimits_{i=1}^{N} \sum\nolimits_{j=1}^{N} a_{ij} b_{z_i z_j}$$

The QAPs have many applications in real world for operational and economic problems, see Loiola, et al. (2007) for a detailed introduction. QAP is known to be a difficult problem to solve, especially when $N > 15$. As stated above, heuristics such as Genetic algorithm (Tate and Smith 1995), Simulated Annealing (Wilhelm and Ward 1987) and Tabu search (Misevicius, 2005) are proposed for retrieving good solutions but the quality of these solutions are unknown and unable to be assessed. Deterministic lower bounds are available for only a few algorithms and rely on parameter choices, e.g., Adams et al., (2007). Derigs (1985) compares Weibull lower bound as representative of SOET with deterministic lower bounds by Branch and Bound algorithm, and concludes that Weibull bounds outperforms deterministic ones. Their research shows the potential of SOET, but the confined experimental design does not provide sufficient support for the usage of SOET, nor suggestions on applications of SOET.

Presumably the usefulness of SOET still applies, as Derigs argued, and it therefore deserves a critical systematic examination of QAPs, and application advice needs to be suggested, which is the focus of this paper. This paper aims at studying the usefulness of SOET with one combination in the SOET framework $\mathcal{J} \times \mathcal{H}^n$, namely the Genetic algorithm on QAPs. The Genetic algorithm is one of the most widely used algorithms in solving operational problems including QAPs (Loiola, et al. 2007). It is known to be able to find good solutions consistently, while being computationally affordable and exceptionally robust to different problem characteristics or implementation (Tate and Smith 1995). It is the leading algorithm that researchers seek to solve QAPs although the quality of the solutions remains ambiguous. Thus it becomes our concern in applying SOETs to see the performance of assessing the quality of Genetic solutions. This paper is organized as follows. In Section 2 we investigate the features of QAPs. Section 3 reviews and proposes methods for statistically estimating the minimum of the objective function as well as the corresponding bounds. Section 4 presents the results and analysis. Section 5 makes a comparison between the SOET lower bounds and the deterministic ones. The last section concludes this paper.

2. Complexity measure of quadratic assignment problem

First we introduce the notations that will be used throughout the paper.

Z_i = feasible solution of a QAP with *N* dimensions, $i = 1,2,\dots,N!$.

Z = the set of all feasible solutions $Z = \{Z_1, Z_2, \dots, Z_{N!}\}$.

$g(Z_i)$ = the value of objective function for solution Z_i.

$\theta = \min_Z g(Z_i)$.

Before going into the comparison of different estimators, the characteristics of the problems need to be investigated. Here we focus on the complexity of the problems which is interpreted here as the difficulty for algorithms to reach θ. In common sense, the complexity of QAPs is decided by the number of dimensions *N*. This makes sense since the number of possible solutions to the QAPs is $N!$ and it determines the size of the population of solutions. Yet, the size of the solution population is not the only effect that influences the complexity of the problems; in fact the structures of matrix *A* and *B* also play an influential role. For example, if one matrix contains most elements equal to 0, the complexity of the problem should be comparably smaller.

Carling and Meng (2015) propose a new way of measuring the complexity of the *p*-median problems in experimental cases. They find that the objective function values for the *p*-median problems are approximately normally distributed, therefore they propose measuring complexity of a problem by the number of standard deviations that the optimal value lies away from the mean, i.e., $\left((\mu_{g(Z)} - \theta)/\sigma_{g(Z)}\right)$, where $\mu_{g(Z)}$ is the mean of $g(Z)$ and $\sigma_{g(Z)}$ the standard deviation. $\mu_{g(Z)}$ and $\sigma_{g(Z)}$ are estimated by drawing a large random sample of solutions. This method provides a good way of measuring the complexity of solving the problems since reaching to θ would grow tough when the it lies further away in the tail; hence the problem is more complex. Although this method is not practically useful since θ is unknown in reality problems, it is quite helpful in assessing the performance of SOETs in experiments. Therefore we follow this way and check the complexity of the QAPs.

The test problems used are from QAPLIB (Burkard et al., 1997). The QAPLIB provides QAP test problems with various N, A and B. One important benefit of QAPLIB is that it has known θ for most problems. We choose 40 problems with N varies between 12 and 100, and then check their complexity.

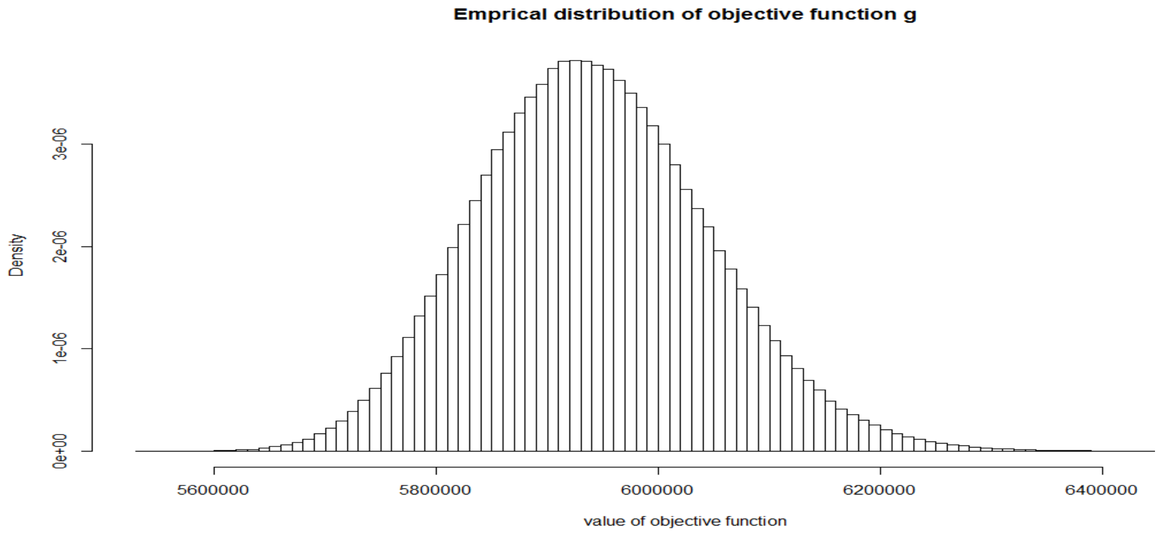

Figure 1: Sample distribution of the 14th problem in the OR-library.

Table 1: Description of the problem complexity of the QAPLIB.

Problem	N	θ	$\mu_{g(z_p)}$	$\sigma_{g(z_p)}$	Complexity
tai64c	64	1855928	2955590.56	346815.9	3.17
bur26c	26	5426795	5941725.45	104825.6	4.91
nug16b	16	1240	1727.95	81.45	5.99
nug24	24	3488	4766.84	159.25	8.03
tai80b	80	818415043	1242911278	30709661	13.82
lipa60b	60	2520135	3272434.13	18091.48	41.58

Figure 1 gives the empirical distribution of a random sample for the problem bur26c. One million random solutions are generated and collected. The value of QAP objective function is approximately normally distributed. The distributions of the other test problems match that in Figure 1. The complexity together with the sample mean and the sample standard deviation for 6 problems are given in Table 1. The full results are given in Appendix I. The complexity of problem varies from 3 to 41. It is easy to reach an optimum which lies only 3 times standard deviation away from the mean, while rather difficult to reach an optimum which lies 41 times standard deviation away from the mean.

3. Statistical estimation of the minimum and its bounds

There are two approaches of SOETs which provide estimators of the minimum and the bounds based on different statistic theories: first, the truncation points approaches, and second, the Extreme value theory approaches. Both approaches require the sample to be randomly selected. However, as Meng and Carling (2014) show, the performance of SOETs requires a randomly selected sample containing values close to θ, in such case, the size of that sample would be enormously large and infeasible to retrieve. As several researchers have pointed out, if the starting values are selected at random, parallel heuristic solutions simulate a random sample in the tail (see e.g. McRoberts, 1971, and Golden and Alt, 1979). In other words, we could get a desired random sample with much less effort. We denote $\tilde{z}_i$ as the heuristic solution in the $i^{th}, i = 1,2,\dots,n$ heuristic process, and use them to compare the functioning of SOETs.

In the truncation points approach, the most commonly used method is the Jackknife estimator introduced by Quenouille (1956):

$$\hat{\theta}_{JK} = \sum_{m=1}^{M+1} (-1)^{(m-1)} \binom{M+1}{m} \tilde{z}_{(m)}$$

where M is the order and $\tilde{z}_{(m)}$ is the m^{th} smallest value in the sample. Dannenbring (1977) and Nydick and Weiss (1988) suggest using the first order, i.e. $M = 1$, for point estimating the minimum. The upper bounds of the JK estimators are the minimum of $\tilde{x}_i$, and the lower bound is $[\hat{\theta}_{JK} - 3\sigma^*(\hat{\theta}_{JK})]$, where $\sigma^*(\hat{\theta}_{JK})$ is the standard deviation of $\hat{\theta}_{JK}$ obtained from bootstrapping the n heuristic solutions (1,000 bootstrap samples are found to be sufficient). The scalar of 3 in computing the lower bound renders the confidence level to be 99.9% under the assumption that the sampling distribution of the JK-estimator is Normal. The 1st order JK-estimator is more biased than the higher order ones, but its mean square error is lower, as shown by Robson and Whitlock (1964). Carling and Meng (2015) checked the performance of 1st and 2nd JK-estimators, finding that 2nd order JK-estimator performs better by providing a higher coverage rate at the cost of a slightly longer interval. The smaller bias of 2nd order JK-estimator improves the performance of the estimator. Therefore, it is reasonable to wonder whether JK-estimators with even higher order would provide better estimation results. To check that, we extend to the 3rd and 4th JK-estimators in our experiments.

The extreme value theory (EVT) approach assumes the heuristic solutions to be extreme values from different random samples, and they follow the Weibull distribution (Derigs, 1985). The confidence interval is derived from the characteristic of Weibull distribution. The estimator for θ is $\tilde{z}_{(1)}$, which is also the upper bound of the confidence interval. The Weibull lower bound is $[\tilde{z}_{(1)} - \hat{b}]$ at a confidence level of $(1 - e^{-n})$, $\hat{b}$ is the estimated shape parameter of the Weibull distribution. Derigs (1985) provides a simple fast way of estimating parameters: $\hat{b} = \tilde{z}_{[0.63(n+1)]} - (\tilde{z}_{(1)}\tilde{z}_{(n)} - \tilde{z}_{(2)}{}^2)/(\tilde{z}_{(1)} + \tilde{z}_{(n)} - 2\tilde{z}_{(2)})$, where $[0.63(n+1)]$ means the integer value of the function.

As stated in the Introduction, Carling and Meng (2014, 2015) argue SOETs would work when $\tilde{x}_{(i)}$ are close enough to θ. They propose a statistic $SR = 1000\sigma(\tilde{z}_i)/\hat{\theta}_{JK}^{(1)}$ to evaluate if that condition is satisfied. That statistic mimics a standardization of the standard deviation for different heuristic solutions. $SR < 4$ indicates the Weibull, and JK intervals cover the optimum almost certainly. It is proved useful in *p*-median problems, and we will check its functioning in QAPs.

4. Experimental evaluation of SOETs

With the SOETs introduced above, we design experiments to investigate their usefulness on Genetic solutions of QAPs. The implementation of Genetic algorithm follows Tate and Smith (1995). The reproduction and mutation proportions are 25% and 75% respectively. The same 40 problems as used for complexity analysis are chosen for experiments. 100 genetic processes with 1000 iterations each are carried out for each problem. With this number of iterations, we have some problems with solutions close to optimum and some far from the optimum, this gives us diversified information for SOET performance in different situations.

The first factor tested is the effect of estimators, where the Weibull estimator together with four JK-estimators are considered. The second factor considered is the effect of n, where we vary n to be 10 and 100. The third factor tested is the effect of complexity. These three factors result in a $40 \times 1 \times 2 \times 5$ experiment combination set, where 40 indicates 40 problem instances in $\mathcal{J}$, 1 indicates 1 heuristic in $\mathcal{H}$, 2 indicates 2 sample size n. To assess the performance of estimators, we first draw a random sample of size *n* with replacement from the 100 solutions, and then calculate the estimators and confidence intervals. The procedure is repeated 1000 times for every combination. Then we get their average relative biasness ($\frac{bias}{\theta} * 100\%$), coverage rate (the proportion of intervals cover θ), and average relative length of the interval ($\frac{length}{\theta} * 100\%$). These three indicators are used to evaluate the performance of the estimators under different circumstances. The performance of SR statistic will also be checked. The results of the experiments are reported below with figures, and the details are provided in Appendix II.

4.1. The relative bias of estimators

First we check the performance regarding the biasness. It is reasonable to expect Jackknife type estimators to have smaller bias than the Weibull estimator. Figure 1 confirms this by giving the Lowess smoothing line (Cleveland, 1979) of the relative bias for the five estimators when sample size n is 10 and 100. But the difference between the four JK-estimators is marginal for both levels of n. When the complexity of the problem is larger than 25, the biases for all the estimators increase sharply. The 2nd and 3rd order JK-estimator have smaller superiority than the 1st and 4th order ones under $n = 10$. The mean difference between 1st and 3rd order JK-estimator is merely 0.23% of θ. When the sample size increases to 100, the advantage of JK-estimators over the Weibull estimator still exists, but the differences between different order JK-estimators diminish. 1st and 2nd order JK-estimators reduce their relative bias by 0.3% and 0.1% of θ respectively, while there is almost no drop for both 3rd and 4th order JK-estimators.

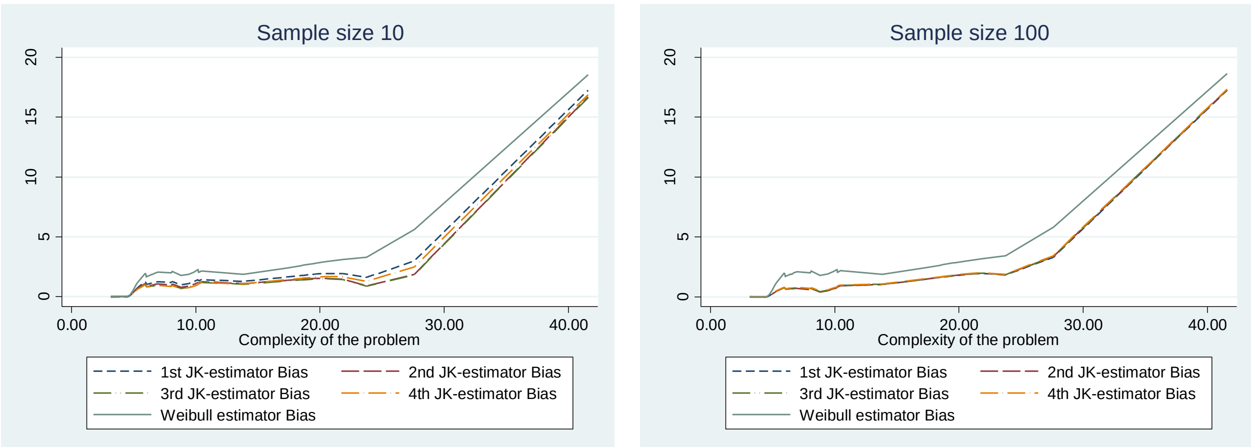

Figure 1. Lowess line of the biasness of the 5 estimators for sample size 10 and 100.

4.2.Interval coverage rate and relative length

Next, we check the coverage rate of the 5 estimators when $n = 10$ and 100. Figure 2 gives the Lowess smoothing line of the results. Table 2 gives the mean and median coverage rate together with lengths. For both levels of n, when the complexity of the problems increases, the coverage rates for all the 5 estimators decline sharply. The JK-estimators again outperform the Weibull ones with a higher coverage rate. Among the JK-estimators, the 4th order performs better than the other three orders except when the complexity goes beyond 30. The 1st order JK-estimator has the worst performance among JK-estimators. Due to the deterioration of 4th order JK-estimator coverage rate, the mean difference between 1st and 4th order JK-estimators is 20% when $n = 10$, and 14% when $n = 100$.

As for the relative length of the interval, Figure 3 shows there is no clear tendency of the relationship between the relative length of the intervals and the complexity of the problems. The Weibull intervals are the shortest, with the mean being around 2.5% of θ for both $n = 10$ and 100. The lengths of the JK-estimator interval almost double when the order increases by 1 and are highly affected by sample size. The lengths when $n = 10$ are almost 3 times of $n = 100$. When $n = 10$, 1st JK-estimator interval has a mean length 3% of θ, a little higher than the Weibull ones while they have the same coverage rate. The 2nd order JK-estimator has a higher coverage rate together with a longer interval. The situation deteriorates for 3rd and 4th JK-estimator with much longer confidence interval. When $n = 100$, the Weibull estimator has almost the same performance while the JK-estimators have better performance. The 3rd order JK-estimator has a similar mean length and a much shorter median length but with 9% more average coverage rate. Thus, when the sample size is small, 1st and 2nd order JK estimators are suggested, otherwise 3rd and 4th order JK-estimators are suggested when the sample size is large.

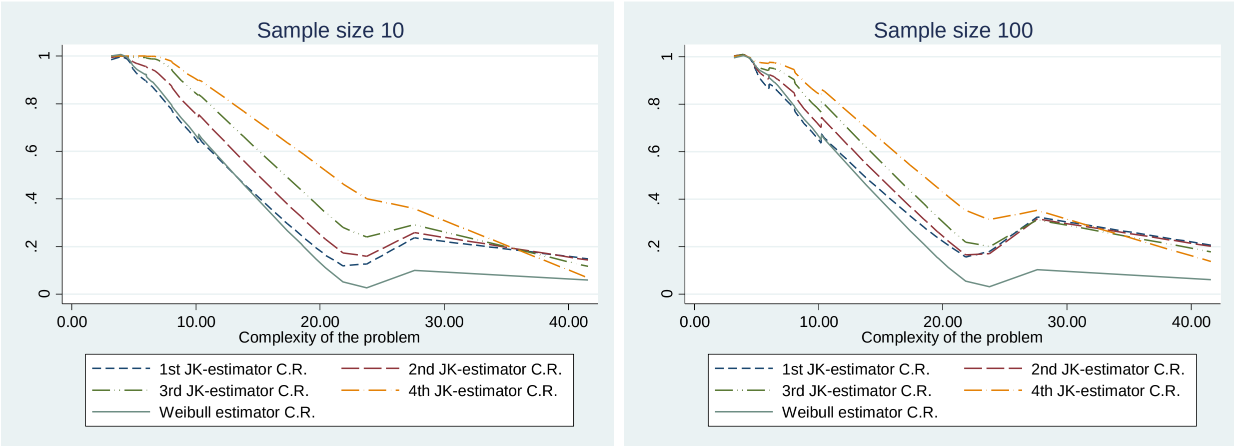

Figure 2. Coverage percentage of 1st order, 2nd, 3rd and 4th order Jackknife estimator, and Weibull estimator under sample size 10 and 100.

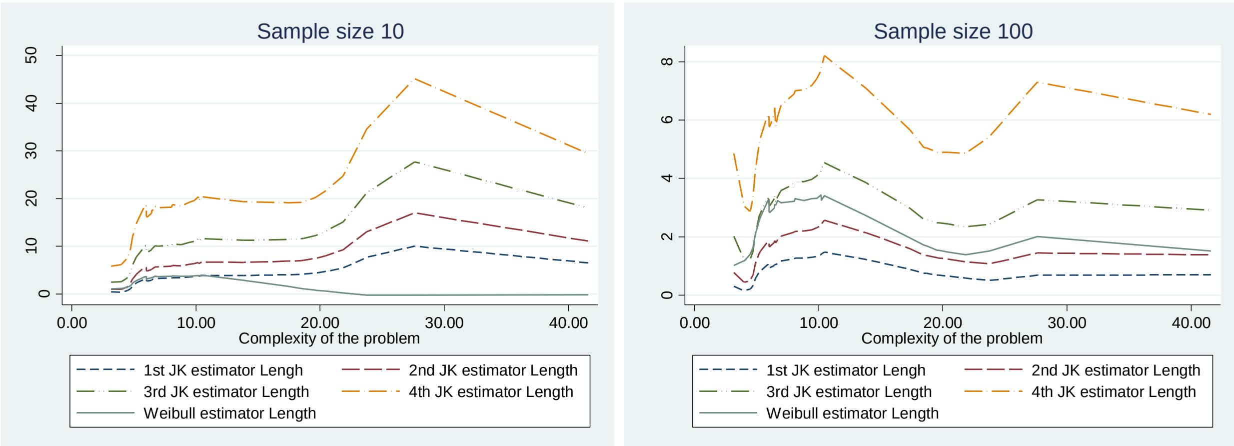

Figure 3. Interval length of 1st order, 2nd, 3rd and 4th order Jackknife estimator, and Weibull estimator under sample size 10 and 100.

Table 2: Coverage rate and relative length 5 estimators.

	Coverage Rate (%)				Relative Length (%)			
	Sample size 10		Sample size 100		Sample size 10		Sample size 100	
Estimator	Mean	Median	Mean	Median	Mean	Median	Mean	Median
1st JK	66	85	67	98	3.01	1.27	0.92	0.14
2nd JK	73	95	71	99	5.24	2.50	1.63	0.24
3rd JK	80	100	75	100	9.18	4.69	2.99	0.45
4th JK	86	100	81	100	16.36	8.74	5.62	0.92
Weibull	66	90	66	90	2.59	1.61	2.48	1.69

4.3.SR performance

Next, the performance of the statistic SR is checked. Based on the analysis above, we focus specifically on 2 cases, the 2nd order JK-estimator when $n = 10$ and 4th order JK-estimator, when $n = 100$. Figure 4 gives the scatter plot and Lowess line between the coverage rate and SR. It can be seen that for both cases, a small SR close to 0 does not guarantee a high coverage rate, while as large a SR as 60 may correspond to a coverage rate even as high as 100%. The problems with high complexities are more likely to have different heuristic solutions trapped in the same suboptimal or similar suboptimals, leading to a trivial SR. As to easy problems, a small SR does indicate a high coverage rate close to 1. Figure 5 provides the instances with SR<7 for both cases. The threshold 7 is chosen because it is the integer part of the smallest SR where the coverage rate for easy problems drops below 0.95 for both sample cases and for all 5 intervals. The size of the circle indicates the complexity of the problem. The problems with trivial coverage rate have complexities over 17. Therefore, the performance of SR is related to the complexity of the problem. For easy problems, small SR supports that the confidence interval covers θ, but not for difficult problems. There is no clear pattern that can be concluded for the functioning of SR. The application of SR remains an open question.

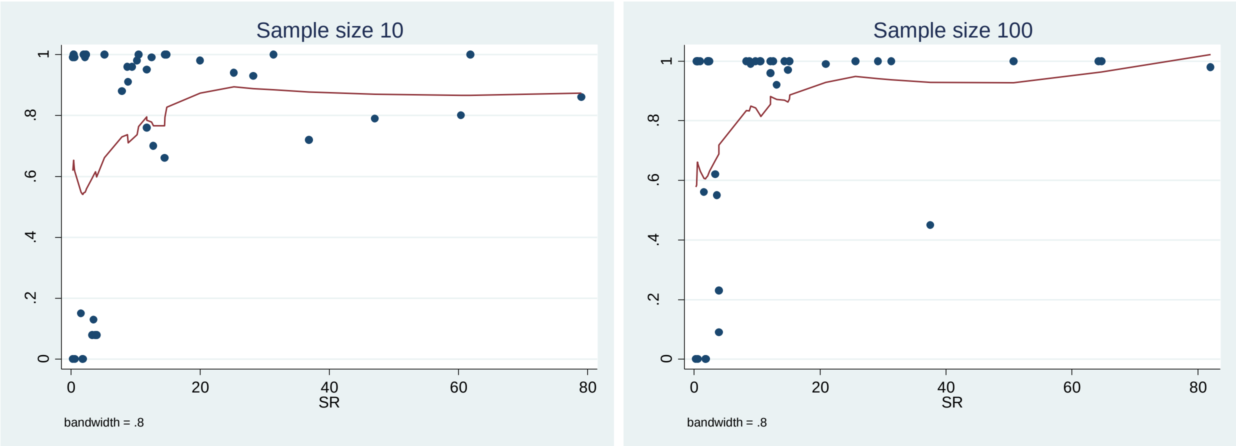

Figure 4. Scatter plot between SR and coverage rate of 2nd order JK-estimator for n=10 and 4th order JK-estimator for n=100.

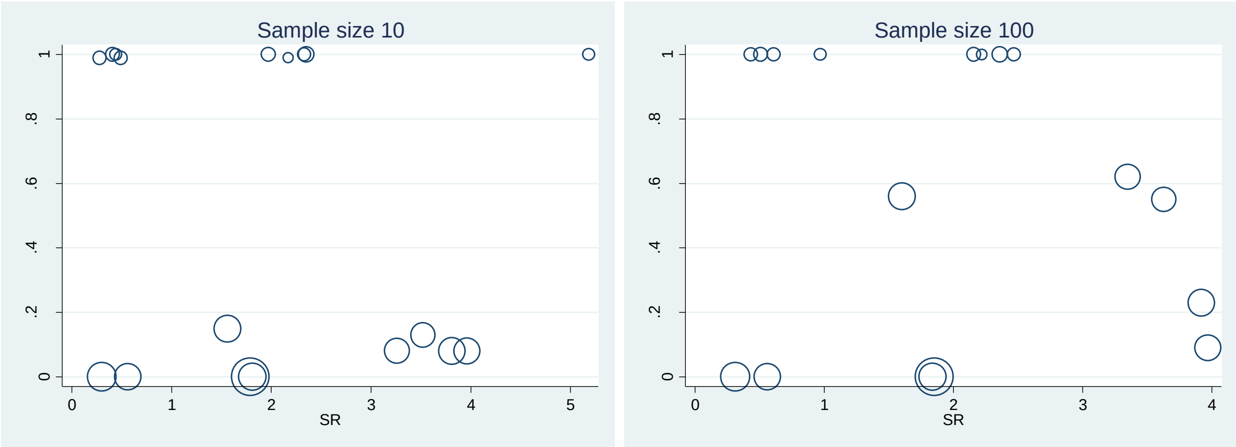

Figure 5. Scatter plot between SR<7 and coverage rate of 2nd order JK-estimator (left), between SR<7 and coverage rate of 4th order JK-estimator (right). The size of the circle stands for complexity of the problem.

5. Lower bound comparison

As a small part in deriving quality of solutions, it is of great concern to compare SOET with the common approach, namely the deterministic bounds, especially the lower bound. Several lower bounds are proposed. Loiola (2007) collects different lower bounds of several problems. The deterministic bounds stated are: Gilmore-Lawler bound (GLB62), from Gilmore (1962); the interior-point bound (RRD95), from Resende et al. (1995); the 1-RLT dual ascent bound (HG98), from Hahn and Grant (1998); the dual-based bound (KCCEB99), from Karisch et al. (1999); the quadratic programming bound (AB01), from Anstreicher and Brixius (2001); the SDP bound (RS03), from Sotirov and Rendl (2003); the lift-and-project SDP bound (BV04), from Burer and Vandenbussche (2006); the Hahn-Hightower 2-RLT dual ascent bound (HH01), from Adams et al. (2007). To incorporate our results to their framework, we compare the SOET lower bound by techniques in the previous section with deterministic ones. The heuristic solutions are derived by running 100 Genetic processes with 3000 iterations each. Almost all the processes stopped improving after 2500 iterations, with very few exceptions. Then we derive the SOET lower bounds by the 4th order JK-estimator. The lower bounds are provided in Table 3. To assess the performance conveniently, we calculate the average absolute relative deviation of the lower bounds, i.e., $\frac{|lower.bound - optimal|}{optimal} * 100\%$, and report them in the last row of Table 3.

Out of 15 problems, SOET has 2 best lower bounds while HH01 and BV04 share the rest 13 best lower bounds. SOET lower bounds perform better than the first 6 deterministic lower bounds but are surpassed by BV04 and HH01, which are acknowledged to be the best deterministic lower bounds. The average absolute bias percentage for SOET, HH01 and BV04 is not large, being 6%, 3% and 3% respectively, yet still significantly smaller than the other 6 lower bounds. Out of 15 problems, SOET lower bounds cover 14 optimums. Therefore, the SOET lower bound is competitive with the best deterministic bounds. It shows great potential in application even though it fails to cover the optimum with small probability.

Table 3: Statistical lower bounds and deterministic lower bounds.

Problem	Optim	GLB62	HG98	KCCEB99	AB01	RS03	BV04	HH01	JK4
Had16	3720	3358	3558	3553	3595	3699	3672	**3720**	3704
Had18	5358	4776	5083	5078	5143	5317	5299	**5358**	5261
Had20	6922	6166	6571	6567	6677	6885	6811	**6922**	6810
Kra30a	88900	68360	75853	75566	68572	77647	**86678**	86247	83867
Kra30b	91420	69065	76562	76235	69021	81156	87699	87107	**88601**
Nug12	578	493	523	521	498	557	568	**578**	515
Nug15	1150	963	1039	1033	1001	1122	1141	**1150**	1143
Nug20	2570	2057	2179	2173	2290	2451	2506	**2508**	2266
Nug30	6124	4539	4793	4785	5365	5803	**5934**	5750	5857
Rou15	354210	298548	323943	323589	303777	333287	**350207**	345210	317782
Rou20	725520	559948	642058	641425	607822	663833	695123	**699390**	679441
Tai20a	703482	580674	617206	616644	585139	6637300	671685	**675870**	656794
Tai25a	1167256	962417	1006749	1005978	983456	1041337	**1112862**	1091653	1084665
Tai30a	1818146	1504688	1566309	1565313	1518059	1652186	**1706875**	1686290	2210730
Tho30	149936	90578	99995	99855	124684	136059	142814	136708	**145616**
Bias.Per		19.08	12.99	13.17	13.65	61.92	2.94	3.26	6.34

Source: Loiola (2007) except for last column and last row. Bold number means best lower bound.

6. Concluding discussion

In this paper, we analyse the performance of SOETs on QAPs. Based on the framework proposed by Giddings (2014), the paper extends the work by Derigs (1985), by systematically verifying the usefulness of SOETs and comparing with deterministic bounds, and it extends the work of Carling and Meng (2014, 2015) by testing on QAPs. We tested 5 estimators on 40 problems with different sample sizes. In our analysis, SOETs can be useful in providing helpful intervals covering the optimum. The JK-estimators have better performance than Weibull estimators. When the sample size is small, the 2nd order JK-estimator is suggested, and when the sample size is large, the 4th order JK-estimator is suggested. The statistics SR do not provide accurate information, especially when the solutions are trapped into suboptimal for complex problems. The comparison between SOET lower bound and deterministic ones shows that the SOET performs close to the best deterministic lower bounds. Thus, it shows SOETs have great potential in accurately assessing the quality of the heuristic solutions.

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Appendix I

Table A1: Description of the 40 problems of the QAPLIB.

Problem	n	θ	$\mu_{g(z_p)}$	$\sigma_{g(z_p)}$	Complexity
els19	19	17212548	58712359.92	10526387	3.94
chr12a	12	9552	45123.58	8833.06	4.03
esc16c	16	160	249.32	20.05	4.45
chr15a	15	9896	61378.34	11444.91	4.5
had12	12	1652	1888.27	50.18	4.71
chr18b	18	1534	4601.64	618.07	4.96
bur26d	26	3821225	4211133.26	75035.14	5.2
nug14	14	1014	1363.54	65.77	5.31
had16	16	3720	4226.97	85.82	5.91
chr20b	20	2298	10708.88	1415.13	5.94
had18	18	5358	5990.01	105.86	5.97
scr20	20	110030	226272.64	19391.22	5.99
chr20a	20	2192	10707.62	1406.16	6.06
nug18	18	1930	2565.41	98.92	6.42
chr25a	25	3796	19877.53	2494.68	6.45
ste36a	36	9526	22750.27	2022.71	6.54
had20	20	6922	7764.79	125.98	6.69
tai35b	35	283315445	516046202	33413753	6.97
nug27	27	5234	7128.04	233.68	8.11
tai40b	40	637250948	1132975203	56180695	8.82
nug30	30	6124	8132.35	212.94	9.43
kra30b	30	91420	137016.96	4621.8	9.87
kra30a	30	88900	134657.56	4487.12	10.2
kra32	32	88900	137137.22	4716.35	10.23
lipa20a	20	3683	3942	24.79	10.45
sko81	81	90998	108443.68	1004.45	17.37
tai60a	60	7208572	8518524.44	70989.41	18.45
sko90	90	115534	136878.06	1143.79	18.66
wil100	100	273038	299759.06	1367.71	19.54
lipa50a	50	62093	64035.91	98.86	19.65
sko100c	100	147862	174507.74	1304.2	20.43
tai80a	80	13557864	15624432.79	94608.26	21.84
lipa70a	70	169755	173757.85	168.43	23.77
lipa40b	40	476581	621324.38	5245.29	27.59

Appendix II

Table A2: Relative bias, coverage rate and relative length of 5 estimators when $n = 10$.

CV stands for coverage rate, RB stands for relative bias in percentage, RL stands for relative length in percentage

Problem	SR	1st JK			2nd JK			3rd JK			4th JK			Weibull		
		CV	RB	RL	CV	RB	RL	CV	RB	RL	CV	RB	RL	CV	RB	RL
bur26d	0.28	0.98	0.00	0.01	0.99	0.00	0.02	1.00	0.00	0.04	1.00	0.00	0.08	0.99	0.00	0.02
lipa70a	0.3	0.00	0.95	0.08	0.00	0.94	0.14	0.00	0.94	0.24	0.01	0.94	0.42	0.00	0.97	0.08
had16	0.41	1.00	0.00	0.03	1.00	0.00	0.06	1.00	0.00	0.15	1.00	0.00	0.34	1.00	0.00	0.04
esc16c	0.44	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.01	1.00	0.00	0.00
bur26c	0.49	0.97	0.00	0.02	0.99	0.00	0.03	1.00	0.00	0.05	1.00	0.00	0.11	1.00	0.00	0.02
lipa50a	0.56	0.00	1.13	0.14	0.00	1.12	0.24	0.03	1.12	0.42	0.17	1.11	0.74	0.01	1.17	0.15
wil100	1.56	0.02	1.59	0.52	0.15	1.53	0.88	0.33	1.48	1.47	0.60	1.45	2.45	0.04	1.72	0.48
lipa60b	1.79	0.00	19.55	0.71	0.00	19.48	1.21	0.00	19.43	2.03	0.00	19.38	3.42	0.00	19.72	0.19
tai80a	1.81	0.00	4.25	0.45	0.00	4.23	0.77	0.00	4.22	1.32	0.05	4.23	2.33	0.00	4.35	-0.15
had18	1.97	1.00	-0.01	0.12	1.00	0.01	0.24	1.00	0.01	0.54	1.00	-0.01	1.19	1.00	0.00	0.20
tai64c	2.17	0.97	-0.02	0.27	0.99	-0.01	0.49	1.00	-0.01	0.93	1.00	0.01	1.83	1.00	0.02	0.37
had12	2.33	1.00	0.00	0.04	1.00	0.00	0.12	1.00	0.00	0.35	1.00	0.02	0.94	1.00	0.00	0.15
had20	2.35	1.00	0.00	0.10	1.00	0.01	0.24	1.00	0.00	0.58	1.00	-0.01	1.36	1.00	0.00	0.26
tai60a	3.26	0.01	3.94	0.99	0.08	3.84	1.69	0.20	3.77	2.83	0.46	3.72	4.75	0.02	4.17	0.84
sko81	3.52	0.03	2.64	0.89	0.13	2.57	1.52	0.35	2.52	2.59	0.70	2.47	4.48	0.04	2.84	0.98
sko9100c	3.81	0.00	3.04	0.94	0.08	2.97	1.61	0.32	2.93	2.74	0.68	2.91	4.73	0.01	3.26	0.91
sko90	3.96	0.00	2.80	0.90	0.08	2.78	1.53	0.34	2.78	2.67	0.80	2.80	4.74	0.02	2.99	0.84
els19	5.18	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.01	1.00	0.00	0.05	1.00	0.00	0.00
lipa20a	7.89	0.88	-0.35	3.75	0.88	-0.46	6.43	0.90	-0.32	10.90	0.95	-0.12	18.70	0.54	0.40	1.59
nug18	8.71	0.86	0.00	1.75	0.96	-0.11	3.02	1.00	-0.18	5.22	1.00	-0.21	9.18	0.93	0.40	1.73
nug30	8.85	0.72	0.47	1.51	0.91	0.41	2.60	0.99	0.37	4.60	1.00	0.39	8.31	0.85	0.78	1.70
nug27	9.51	0.88	-0.16	2.22	0.96	-0.23	3.80	1.00	-0.22	6.59	1.00	-0.18	11.59	0.90	0.33	2.28
nug24	10.22	0.93	-0.14	2.04	0.98	-0.17	3.52	1.00	-0.17	6.25	1.00	-0.17	11.34	0.97	0.24	2.19
nug14	10.48	0.99	-0.10	1.31	1.00	-0.03	2.41	1.00	0.00	4.77	1.00	0.03	9.54	1.00	0.08	1.76
kra30b	11.7	0.84	0.22	2.33	0.95	0.11	4.01	0.99	0.07	7.00	1.00	0.09	12.43	0.92	0.71	2.45
kra32	11.73	0.53	1.21	2.74	0.76	0.96	4.71	0.95	0.77	8.08	1.00	0.62	14.00	0.59	1.79	2.97
nug16b	12.46	0.99	-0.06	1.65	0.99	0.15	3.41	1.00	0.20	7.52	1.00	0.18	16.11	0.96	0.01	1.89
kra30a	12.77	0.40	1.28	3.26	0.70	0.95	5.62	0.95	0.68	9.55	0.99	0.46	16.34	0.41	2.05	3.83
tai80b	14.51	0.43	2.60	3.23	0.66	2.38	5.55	0.91	2.21	9.61	0.99	2.06	16.89	0.39	3.31	3.40
chr18b	14.54	1.00	-0.03	0.73	1.00	0.05	1.56	1.00	0.02	3.64	1.00	0.03	8.22	1.00	0.00	1.64
scr20	14.85	0.99	-0.17	1.93	1.00	-0.03	3.52	1.00	0.05	6.92	1.00	0.11	13.73	0.99	0.06	2.42
tai35b	20.02	0.94	0.18	1.73	0.98	0.23	3.05	1.00	0.30	5.72	1.00	0.39	11.11	0.98	0.47	2.33
ste36a	25.24	0.81	1.19	4.81	0.94	0.98	8.31	1.00	0.83	14.68	1.00	0.73	26.50	0.88	2.20	4.75
tai40b	28.25	0.83	0.33	6.37	0.93	-0.09	10.85	0.98	-0.26	18.47	1.00	-0.25	31.94	0.90	1.68	11.50
chr12a	31.36	1.00	-0.01	1.23	1.00	0.06	3.13	1.00	0.00	8.07	1.00	0.14	19.43	1.00	0.00	3.43
chr20b	36.85	0.36	6.58	6.92	0.72	6.52	12.03	0.95	6.57	21.72	1.00	6.77	40.20	0.50	7.83	6.78

lipa40b	47.08	0.74	-0.79	28.43	0.79	-3.25	48.22	0.82	-3.32	78.48	0.84	-1.87	127.79	0.33	5.04	-0.74
chr20a	60.44	0.61	3.76	15.68	0.80	2.09	26.80	0.96	0.90	45.06	1.00	0.18	76.18	0.63	7.20	15.66
chr15a	61.88	1.00	-0.09	2.18	1.00	0.14	4.61	1.00	0.05	10.83	1.00	-0.01	24.74	1.00	0.03	4.81
chr25a	79.02	0.68	8.01	18.56	0.86	7.16	31.68	0.97	7.00	54.64	1.00	7.25	96.12	0.75	11.82	19.80

Table A3: Relative bias, coverage rate and relative length of 5 estimators when $n = 10$.

CV stands for coverage rate, RB stands for relative bias in percentage, RL stands for relative length in percentage

Problem	SR	1st JK			2nd JK			3rd JK			4th JK			Weibull		
		CV	RB	RL	CV	RB	RL	CV	RB	RL	CV	RB	RL	CV	RB	RL
bur26d	0.31	0.00	0.94	0.03	0.00	0.94	0.05	0.00	0.94	0.09	0.00	0.94	0.16	0.01	0.97	0.08
lipa70a	0.43	0.98	0.00	0.00	0.98	0.00	0.00	0.99	0.00	0.00	1.00	0.00	0.01	0.99	0.00	0.01
had16	0.51	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.05
esc16c	0.56	0.00	1.09	0.10	0.00	1.09	0.18	0.00	1.09	0.31	0.00	1.10	0.55	0.01	1.17	0.15
bur26c	0.61	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.02
lipa50a	0.97	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00
wil100	1.6	0.00	1.45	0.33	0.02	1.45	0.55	0.12	1.45	0.95	0.56	1.46	1.68	0.05	1.72	0.31
lipa60b	1.84	0.00	4.22	0.11	0.00	4.22	0.19	0.00	4.22	0.36	0.00	4.22	0.69	0.00	4.35	0.31
tai80a	1.85	0.00	19.35	0.49	0.00	19.37	0.85	0.00	19.39	1.54	0.00	19.38	2.85	0.00	19.72	0.56
had18	2.16	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.21
tai64c	2.22	1.00	0.00	0.00	1.00	0.00	0.01	1.00	0.00	0.02	1.00	0.00	0.06	0.99	0.02	0.37
had12	2.36	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.27
had20	2.47	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.14
tai60a	3.35	0.00	3.55	0.83	0.00	3.51	1.41	0.14	3.49	2.39	0.62	3.48	4.10	0.03	4.18	0.86
sko81	3.63	0.00	2.34	0.63	0.02	2.31	1.07	0.38	2.30	1.81	0.55	2.30	3.09	0.03	2.83	0.82
sko9100c	3.92	0.00	2.85	0.44	0.00	2.84	0.74	0.01	2.85	1.29	0.23	2.88	2.27	0.02	3.24	0.86
sko90	3.97	0.00	2.70	0.32	0.00	2.71	0.55	0.00	2.72	0.97	0.09	2.74	1.75	0.01	2.99	0.89
els19	8.31	1.00	0.00	0.02	1.00	0.00	0.07	1.00	0.00	0.22	1.00	0.00	0.63	0.53	0.34	1.77
lipa20a	8.75	0.99	-0.02	0.37	0.99	0.01	0.72	1.00	0.02	1.47	1.00	0.01	3.00	0.92	0.41	1.71
nug18	9.02	0.44	0.35	0.37	0.76	0.35	0.68	0.97	0.35	1.28	0.99	0.36	2.46	0.85	0.78	1.66
nug30	9.76	1.00	0.00	0.05	1.00	0.00	0.13	1.00	0.00	0.33	1.00	0.00	0.82	0.92	0.28	2.39
nug27	10.46	1.00	0.00	0.07	1.00	0.00	0.14	1.00	0.00	0.33	1.00	0.00	0.74	0.97	0.27	2.10
nug24	10.62	1.00	0.00	0.01	1.00	0.00	0.03	1.00	0.00	0.08	1.00	0.00	0.22	1.00	0.08	1.76
nug14	12.13	0.81	-0.34	3.25	0.89	-0.54	5.46	0.95	-0.54	8.99	0.96	-0.46	14.96	0.56	1.80	2.64
kra30b	12.15	0.80	0.14	0.40	0.98	0.17	0.72	1.00	0.19	1.37	1.00	0.21	2.65	0.90	0.72	2.35
kra32	12.58	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	0.97	0.01	1.96
nug16b	13.12	0.85	-0.23	3.86	0.85	-0.16	6.70	0.86	0.11	11.76	0.92	0.37	20.99	0.41	2.09	3.56
kra30a	14.35	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.01
tai80b	14.89	0.40	1.83	1.84	0.65	1.76	3.17	0.86	1.76	5.47	0.97	1.83	9.56	0.38	3.35	4.16
chr18b	15.14	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	1.63
scr20	15.17	1.00	0.00	0.01	1.00	0.00	0.02	1.00	0.00	0.04	1.00	0.00	0.10	0.99	0.07	2.47
tai35b	20.9	0.20	0.22	0.17	0.56	0.21	0.29	0.91	0.21	0.54	0.99	0.21	1.02	0.98	0.47	2.27
ste36a	25.61	0.74	0.74	1.62	0.87	0.79	2.82	0.97	0.87	5.05	1.00	0.92	9.26	0.89	2.24	5.51
tai40b	29.2	1.00	-0.01	1.27	1.00	0.07	2.36	1.00	0.11	4.74	1.00	0.10	9.61	0.90	1.69	6.23

chr12a	31.36	1.00	-0.01	1.23	1.00	0.06	3.13	1.00	0.00	8.07	1.00	0.14	19.43	1.00	0.00	3.25
chr20b	37.55	0.00	6.36	1.11	0.00	6.41	1.94	0.07	6.47	3.60	0.45	6.53	6.86	0.48	7.98	8.14
lipa40b	50.75	1.00	-0.04	1.71	1.00	0.08	3.73	1.00	0.12	8.61	1.00	0.25	19.53	0.33	5.51	5.50
chr20a	64.2	0.96	-0.52	8.63	0.99	-0.49	14.91	1.00	-0.32	26.65	1.00	-0.14	48.59	0.65	7.33	9.11
chr15a	64.72	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.00	0.00	1.00	0.04	5.05
chr25a	81.99	0.54	5.17	7.38	0.77	4.93	12.49	0.91	4.99	21.35	0.98	5.16	37.22	0.72	12.12	17.97