MATHEMATICAL REASONING AT PRE-SCHOOL LEVEL

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In this paper, young children's mathematical reasoning is explored from the standpoint of two different types of frameworks. I analyse and discuss two cases of reasoning emphasising the use of mathematical foundation in arguments and choices that children make when solving mathematical problems. The first framework focuses on arguments and warrants. The other helps us to categorise different types of reasoning. In both frameworks, the mathematical foundation is central.

INTRODUCTION

I'm sitting on the train. My seat is one of four sharing a table. The other three seats are occupied by a mum and two small children, a boy and a girl. The boy, who is the oldest of the two, turns to me and says: "I'm four!". I smile, and ask the little girl how old is she is. "I'm four!", she replies. The boy laughs and says: "No, she is two!" and shows me two fingers to illustrate. "Ok. So you are four and your sister is two. How much older are you than your sister?", I ask the boy. He looks at me a bit puzzled. Then he takes up four fingers on his left hand, and two fingers on his right and places the hands opposite each other so he can compare the number of fingers. I can see him nodding when he is counting the fingers on his left hand that doesn't meet any finger from the right hand. One nod. One more nod. "Two!", he says with a smile. The mother looks at me and says, "I have never seen him doing that before." The boy turns to me again: "You are a big girl, aren't you?"

Research has shown that young children are more capable of developing mathematical concepts and processes than previously thought (Clements & Sarama, 2007; Mulligan & Vergnaud, 2006). This is further emphasised by studies focusing on general mathematical processes such as problem solving, argumentation and justification (Perry & Dockett, 2008), early algebraic reasoning (Papic & Mulligan, 2011), and modelling and statistical reasoning (English, 2012). Recent Swedish research shows how young children can use different mathematical competencies in their mathematical reasoning (Säfström, 2013). The children were able to question other children's arguments and to justify their own ones. This type of negotiation is a central part as pattern of interaction when creating a collective mathematical reasoning (c.f. Voigt, 1994).

Although the development of process and sense making of mathematical concepts can take place without explicit guiding (McMullen, Hannula-Sourmunen & Lehtinen, 2013), there seems also to be evidence that children that are not stimulated to train processes of this particular kind do not develop such competencies (Bobis, et al, 2005; Bobis, Mulligan & Lowrie, 2008). It appears to be more about the situation and the opportunity to learn (Hiebert, 2003), and what we learn is a result of the type of activities that we participate in. A body of research concludes that with the guidance from an adult, the mathematical ideas can be more extensive and explicitly examined (Björklund, 2008; Lee & Ginsburg, 2009; van Oers, 1996). The boy on the train met a mathematical problem in steps, a result of guide (a 'big girl') who knew how to break up a problem in sub-tasks, but the solution strategy was a product of his own creativity. In this paper, I'll explore young children's mathematical reasoning to highlight some variations of reasoning, and illustrate these with two examples.

MATHEMATICAL REASONING

Mathematical reasoning is a part of several frameworks (e.g. NCTM, 2000; Niss, 2003) and we find it in the Swedish curriculum from pre-school level all the way up to upper secondary school level (School Agency, 2011a; 2011b; 2011c). One of the goals that Swedish preschools should aim for is that children "develop their mathematical skill in putting forward and following reasoning" (School Agency, 2011a, p. 10). This is a rather challenging goal especially considering that Swedish children struggle with mathematical reasoning and problem solving later, for instance in international test such as TIMSS (Skolverket, 2012). Despite this central role, there are not many frameworks that aim to characterise reasoning in detail (Lithner, 2008). Most often, the concept reasoning is used to describe a reasoning of high quality that is seldom defined.

For young children, mathematical reasoning is often related to oral language skills (Charlesworth, 2005). One way of studying reasoning is to look at argumentation. We can, for instance, look at individual's argumentation and different choices made when solving tasks e.g. Lithner (2008) or Sumpter (2013). This is fruitful when you want to study different types of reasoning produced by individuals in task solving such as imitative reasoning and creative mathematical founded reasoning (Lithner, 2008). However, it is plausible to expect other forms of reasoning generated when pre-school children are trying to solve mathematical tasks and exercises, mainly because of different contexts where activities are taking place. Therefore, in this paper I'll use two different frameworks to discuss two cases of reasoning.

Individual reasoning

When you want to focus on individual's reasoning and arguments for choices they make in different stages it is helpful to have a framework with a clear definition of reasoning and a clear structure to organise data. Here we use the framework provided by Lithner (2008). In this framework, reasoning is defined as the line of thought adopted to produce assertions and reach conclusions in task solving. This means that this line of thought does not have to be based on formal logic; it could even be incorrect. This is product that starts with a task and ends with some sort of answer. To structure the data, reasoning is seen as four steps: (1) A problematic situation (PS) is met where it is not obvious for the individual how to proceed; (2) A strategy choice (SC) is made, a choice that can be supported by a predictive argument; (3) The strategy is implemented (SI) and the implementation can be supported by verifying arguments; and, (4) A conclusion (C) is obtained. This is not necessarily a linear structure; the individual can jump between the steps in his or her reasoning. The argumentation is the part of reasoning that aim to convince you (or someone else) that the reasoning is appropriate. 'Choice' should not only been seen as a result of a conscious decision. It includes actions that are more subconscious.

An important part of this framework is the content of an argument. When focusing on the content, Lithner (2008) uses the notion of anchoring. The argument is anchored by the individual in relevant mathematical properties of the components in the reasoning. These components are objects, transformations, and concepts. Objects are fundamental entities, e.g. numbers, variables, and functions. A transformation is a process to an object where a sequence of these transformations is a procedure, e.g. finding a polynomial maxima. Concepts are central mathematical ideas built on a set of objects, transformations, and their properties, e.g. infinity concept. Some properties are more relevant than others, and the division of surface and intrinsic properties aim to capture what is relevant and not. This is dependent of the context:

> "In deciding if 9/15 or 2/3 is largest, the size of the numbers (9, 15, 2, 3) is a *surface* property that is insufficient to resolve the problem, while the quotient captures the *intrinsic* property" (Lithner, 2008, p.261).

When the boy on the train solved the problem 4 – 2, he used comparison which is a transformation to the objects 'cardinal number 4' and 'cardinal number 2'. He could also have used subtraction in the meaning of take away, for instance counting down from 4 to 2. This would have been another transformation to the same objects. In this way, we can see which arguments the individual has for various strategy choices and conclusions.

In this framework, we separate creative and imitative mathematical reasoning. Reasoning is defined as *Creative Mathematically Founded Reasoning* (CMR) if it fulfils following conditions: (1) Novelty; (2) Plausibility; and, (3) Mathematical Foundation. It is important to stress that creative mathematical thinking is not restricted to people with an exceptional ability in mathematics, but it can be hard to perform without the right competencies. Therefore students might not even try to produce a CMR (Lithner, 2008) even in situation when they easily could have made progress (Sumpter, 2013). The competencies are knowledge (the mathematical foundation), heuristics, beliefs and control (Schoenfeld, 1985). They are both cognitive (e.g. the mathematical knowledge) and affective (beliefs).

Imitative reasoning is a family of different types reasoning: *Memorised Reasoning* (MR) where the strategy choice is founded on recalling an answer and the strategy implementation consists of writing this answer down with no other consideration; and, *Algorithmic Reasoning* (AR) where the strategy choice is recalling a certain algorithm (set of rules) that will probably solve the problematic situation. Algorithmic Reasoning has three sub-categories: Familiar AR, Delimiting AR and Guided AR. (For a longer discussion and further explanations, see Lithner, 2008.)

Collective reasoning

Another way of looking at reasoning, especially when it is a result of social interaction, is to see it as a collective process. Here, we turn to Krummheuer (2007) who studies students learning mathematics through participation in processes of collective argumentation. This process is social and "comprises a set of practices and norms that are collective" (Ball & Bass, 2003, p. 29). One of the strengths of this framework is the notion of argument with the division of data, conclusion, warrant, and backing. It also allows us to see how arguments are used, for instance how they are directed. Previous research has shown that during free outdoor play, Swedish pre-school children use a variation a products and procedures in their argumentation when they challenge, support and take the reasoning forward (Sumpter & Hedefalk, forthcoming). When needed to, they use concrete materials to strengthen their arguments and also as an aid in order to reach conclusion, but also abstract social constructs such as jokes as part of their reasoning.

In an analysis of argumentation (AA), Krummheuer (2007) suggests the use of warrant and backing. The notions of data, conclusion, warrant and backing come from Toulmin (2003). Warrant can be defined as the statements that add to legitimize the reasoning. Backings are about what are permitted representing "unquestionable basic convictions (Krummheuer, 2007, p. 65). Together, arguments can be linked to each other creating a chain, a reasoning sequence. An accepted conclusion can work as data for a new argument. Just as in Sumpter and Hedefalk (forthcoming), here I would like to focus on the arguments from a mathematical point of view. Therefore, I use Krummheuer's (2007) structure and notions together with Lithner's (2008) concept of anchoring arguments in mathematical properties when studying the conclusion, warrants, and backing. (For a longer discussion and further explanations e.g. analysis of participation, see Krummheuer, 2007.)

TOM AND JIM

This observation comes from a set of pilot studies. The author spent a few days (different times over a period of two years) in a nursery as an observer taking notes about children's interactions. This episode comes from an occasion when the nursery took the five-year olds to play in the woods. Tom and Jim (both 5 years) are playing with sticks. The sticks are in their game laser swords and the question, brought up by the boys themselves, is: Which stick is the longest?

	Data	Argument	Conclusion
Tom	Look! A laser sword! [picks up a stick from the ground]		
Jim	I got one, too. [picks up a stick from the ground]		
[Tom and Jim are playing with the sticks for a few minutes]			
Tom	My sword is long	The length of Tom's stick is large.	
Jim	My sword is longer.	The length of Jim's stick is longer than the length of Tom's stick.	B > A
Tom	No, it is not!	Objection to B > A.	
Jim	Look! [holds up his stick next to Tom's stick]	Comparing magnitudes, here lengths. The length of Jim's stick is longer than the length of Tom's stick.	Since my stick is longer than yours, B > A.
Tom	Ok. But mine is thicker. Boom boom! [Pretend shooting]		B > A
Jim [looking around for other sticks]	Look at this one then! [Drags out a large branch]		
Tom	That one is longer than mine! That one is a laser	The branch is longer than Tom's stick. This	C > A

	cannon!	is concluded without a direct measure. This requires an understanding of conservation of length.	
Jim	It is the longest. Ha ha! It is a cannon!	Possible use of transitivity: C > B > A	C > B, C > A

Table 1: Tom and Jim solving "Which stick is the longest?"

Tom and Jim use several mathematical properties concerning measurement when taking their reasoning forward. They compare magnitudes and order, they use conservation and there is a possible use of transitivity when working with three objects. These function as a mathematical foundation; as warrants they are the ground of the arguments. The arguments are directed and there are elements of challenge ("No, it is not!") and support ("It is the longest."). Tom and Jim arrive at a conclusion they both agree on.

HEIDI

This episode comes from a set of video films. The author asked parents to children from three pre-schools if their children could participate in a problem solving session. The firsts ones to agree were Heidi's parents. This recording was made in their home simply to avoid having to get the permissions to record at the pre-school. Since the focus is on Heidi's reasoning and not on the social context, the assumption is that this episode although in a some sort of lab-setting still works as an illustration of a pre-school child's mathematical reasoning. Heidi is 3 years and 5 months old. As an introduction, she has solved three tasks (is this with the blocks, or in her head?): one task basic addition (4 + 3); one task basic subtraction (5 – 2) and one task basic division (4/2). The interviewer then tried to give all the blocks to one teddy. Heidi objected to this. She is now trying to solve the following task: 9 blocks should be divided by three teddies (Teddy Bluebear, Rabbit-y and George the Dog). How many blocks do they get each? The blocks consist of three colours (three of each), green, blue, and yellow, but are all mixed up.

Interviewer: Shall we see if George the Dog can count the blocks?

Heidi: Yes. [Counts when the interviewer points to the blocks one by one with George the Dog's paw] One, two, three, four, five, six, seven, eight, nine. Nine!

Interviewer: Nine blocks! Look, George the Dog is really happy!

Heidi: [laughs] Why is he shaking?

Interviewer: He is happy! That is what George the Dog is doing when he is happy. Shall we divide the blocks? Shall we divide so they get a few blocks each? Do you want to do that?

Heidi: Yes.

Interviewer: Shall we do it together? Who should have this one? [Points at a yellow brick that is closest to the interviewer.]

Heidi: Rabbit-y!

Interviewer: Then…?

Heidi: [Points at another yellow block]

Interviewer: Who should have this one?

Heidi: Bluebear. [points at the remaining yellow block]

Interviewer: Who should have this yellow block?

Heidi: The dog.

Interviewer: What should we do know?

Heidi: The green and the blue ones.

Heidi distributes first the green blocks and then the blue blocks to the teddies.

Reasoning structure

The data is organised using the reasoning structure suggested by Lithner (2008): Problematic Situation (PS); Strategy Choice (SC); Strategy Implementation (SI); and, Conclusion (C).

PS: Nine blocks should be divided by three teddies.

SC: Identify property of the blocks: three colours. Group the blocks after colour: 9 = 3 + 3 + 3. Then perform division: 9/3 = (3 + 3 + 3)/3 = 3/3 + 3/3 + 3/3.

SI: Straightforward. First yellow, second green, last blue.

C: Each teddy gets 1 blue, 1 green and 1 yellow resulting in 3 blocks.

In this sequence, although the interviewer asks questions, Heidi is the one making all the central decisions. She decides which blocks are going to be shared out (except for the first one), in which order they should be shared and how many at a time. Heidi recognises the colour of the blocks and uses this property when grouping the blocks into smaller sub-sets. Then she performs division for each of these sub-sets. The task is considered being a new problem for Heidi and her reasoning is novel. Her choice to group the blocks is both plausible and has a mathematical foundation. This reasoning is categorised as *Creative Mathematically Founded Reasoning* (CMR).

DISCUSSION

In this paper we have seen two cases of reasoning described and analysed using two different types of frameworks. In the case of Tom and Jim, Krummheuer's (2007) framework brought forward content and direction of arguments. The mathematical foundation, here properties of measurement, helped them to reach a conclusion. Similar behaviour has been observed in previous studies (e.g. Sumpter & Hedefalk, forthcoming). Just as in Säfström (2013), Tom and Jim show that they can use different mathematical competencies and the ability to challenge and justify arguments. In the case of Heidi, we can see another type of reasoning. Even though she is interacting with the interviewer (mainly through a teddy), she makes all the central decisions. The analysis of strategy choice and conclusion makes her reasoning categorised as Creative Mathematically Founded Reasoning (CMR). The two different frameworks brought forward different aspects of reasoning, each of them suitable to the different types of data. In the case of collective reasoning, directions of arguments are helpful in order to understand the social process. In the case of individual problem solving, focusing on strategy choice and conclusion allow us to categorize different types of reasoning.

Here we had a case of CMR, a type of reasoning that is helpful when you want to solve a mathematical problem and you don't know a specific solution method. However, it would have been rather surprising if Heidi performed an Imitative Reasoning considering that she has not met so many mathematical procedures (if any at all). A possible theory is that Imitative Reasoning is something you can produce when you have access to a lot of mathematical procedures and you have met a lot of mathematical tasks where most of the tasks are linked to a specific procedure, especially when working alone with a textbook (Lithner, 2008). Heidi has not experienced this yet. Most likely, she doesn't have a belief saying that certain task should be solved with a certain algorithm (c.f. Schoenfeld, 1985). Her reasoning, at the moment, is only limited to her mathematical knowledge, her creativity and the milieu.

In the curriculum for Swedish pre-school, we can read that preschool teachers are responsible for work in the group of children taking place so that children "are stimulated and challenged in their mathematical development" (School agency, 2011a, p. 11). Moreover, research has indicated that students learn what they have opportunity to learn (Bobis, et al, 2005; Bobis, Mulligan & Lowrie, 2008; Hiebert, 2003) and that young children that have the guidance from an adult can expand their mathematical thinking even further than without a guide (Björklund, 2008; Lee & Ginsburg, 2009; van Oers, 1996). Given all this, it would be interesting to see what Tom, Jim and Heidi could do if their reasoning was stimulated. What reasoning could they perform? Tom, Jim and Heidi show creativity and skills for putting arguments based on mathematical properties forward. Their ability is recognised, the question is what are we doing with it?

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