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CONSISTENT METHOD OF MOMENTS ESTIMATION OF THE TRUE FIXED EFFECTS MODEL

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Abstract. A couple a years ago William H. Greene introduced the so called 'True fixed effects estimator' (TFE), which is intended to separate between heterogeneity and efficiency in stochastic frontier analysis. We would say that it has had huge impact on applied stochastic frontier analysis. One problem with the original TFE estimator, is that it is biased in cases with finite time observations. For the normal-half-normal model this problem was solved by Chen et al. (2014) based on maximum likelihood estimation of the within-transformed model. In this study we show the possibilities with method of moment estimation. This approach is more straightforward computational and is more flexible than maximum likelihood estimation since the estimators are not as dependent on the distributional assumptions and do not hinges on an explicit distribution of the random error. We only assume symmetry and for more complicated models also a fixed fourth order cumulant.

Key words and phrases. Stochastic frontier, Fixed effects, Panel data, Method of moments.

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1. Introduction

Greene introduced the so called 'True random effects estimator' (TRE) and the 'True fixed effects estimator' (TFE), which are intended to separate between heterogeneity and efficiency in stochastic frontier analysis (Greene 2005a,b). However, the proposed TFE estimator was biased and inconsistent in cases with finite time observations. The firm specific inefficiency estimator is usually inconsistent in this case but for the TFE estimator also the estimator of the parameter of the frontier function is inconsistent.

Wang & Ho (2010) proposed a consistent estimator of a similar stochastic frontier model as Greene had considered. The difference is that the inefficiency term in Wang & Ho (2010) model is only allowed to vary over time with observable variables included by the researcher. The model proposed by Greene varies over time due unobserved factors.

Chen et al. (2014) proposes a consistent estimator based on Greene's model and they derive the distribution of the composed error for the within transformed model where the fixed effects have been removed. Based on the within transformed model Chen et al. (2014) derives a likelihood function which does not depend on the fixed effects and therefore argue their estimator does not suffer from the incidental parameters problem. Their estimator should be consistent and asymptotic normal, which they conclude with a statement "that the MLE based on a correctly specified likelihood is consistent, for "regular" problems". No formal proof is conducted but Monte Carlo simulations are provided that support their case.

We admire the effort Chen et al. (2014) have given proof of by actually deriving a likelihood function of the within transformation of the normal-half-normal TFE model proposed by Greene. However, in this study we develop a more straightforward way to derive consistent estimators of classes of frontier models. Based on method of moments methodology it is relatively simple to derive estimators for any TFE estimator with one-parameter distributed efficiency term and symmetric random error. If one also add a fixed cumulant of order four of the random error, then modelling with a two-parameter distributed efficiency term is possible. Thus, this e.g. enables the normal-gamma stochastic frontier model.

2. Stochastic frontier Model

In this paper we consider the following stochastic frontier model:

(1)
$$y_{it} = \alpha_i + x_{it}\beta + \epsilon_{it}, \quad \epsilon_{it} = \nu_{it} - u_{it}, \quad u_{it} \geq 0, \quad t = 1, 2 \ldots, T \ \& \ i = 1, 2 \ldots, N,$$

where y_{it} is a single output (possibly log-transformed), x_{it} is a $K \times 1$ of inputs, β is a $K \times 1$ coefficient vector and α_i is the firm effect of firm i, u_{it} is the measure of technical inefficiency of firm i at time t and ν_{it} is a random error term. We call ϵ_{it}

the 'composed error' term. The ν_{it} are assumed to be iid as well as u_{it} and x, ν and u are mutually independent. No assumptions are made about α_i.

3. Example of two Stochastic frontier Models

In this section we propose two stochastic frontier models that can be handled with appropriate sample moments estimators. First we consider the original normal-half-normal model proposed by Greene (2005*a*,*b*) and second we consider the normal-gamma model considered in cross-sectional modelling by Greene (1990), but as we know not before in a panel data setting. In the next section we present consistent moment estimators to consistently estimate the parameters of the models in this section.

The normal-half-normal model consider the inefficiency term to be half-normally distributed and the random error to be normally distributed. To obtain a consistent estimatot of the parameter of the half-normal distribution it is enough to assume that the random error is symmetrically distributed, i.e. that the third central moment is zero.[1]. The half-normal distribution is a one-parameter distribution and we define the parameter as follows:

$$\theta = \sqrt{\frac{\pi - 2}{2}} \frac{1}{\sigma_u}$$

where $u_{it} \sim |N(0, \sigma_u^2)|$. Given that the random error is symmetric the third order central moment of the composed error term is equal to the negative of the third order central moment of the inefficiency term:

$$\mu_{3,\epsilon} = \mu_{3,\nu} - \mu_{3,u} = -\mu_{3,u}.$$

The third order central moment of a half-normally distributed random variable is:

$$-\mu_{3,\epsilon} = \frac{4 - \pi}{2\theta^3},$$

which is strictly positive since θ is, and an expression of θ can then be shown to be:

$$\theta = \left(\frac{4 - \pi}{-2\mu_{3,\epsilon}} \right)^{1/3}. \tag{2}$$

Thus if we have a consistent estimator of $\mu_{3,\epsilon}$ we can consistently estimate θ. Given a consistent estimator of θ and of $\mu_{2,\epsilon}$ one can in turn obtain consistent estimators of the expected value and the variance of the random error ν_{it} from the following expressions:

[1]However, to obtain firm specific predictions of the inefficiencies one need to assume normality following Jondrow et al. (1982).

$$E(\epsilon_{it}) = -E(u_{it}) + E(\nu_{it}) = 0, \tag{3}$$

$$\mu_{2,\epsilon} = V(u_{it}) + V(\nu_{it}). \tag{4}$$

This enables to obtain firm specific predictions of the inefficiencies given the normal assumption on ν_{it} (see eq. (6) Greene 2005*b*).

Now the second case where we assume $u_{it} \sim gamma(\alpha, \beta)$ and $\nu_{it} \sim N(\mu_\nu, \sigma_\nu^2)$. For ν_{it} one could select any distribution with constant fourth order cumulant. The normality assumption is standard in the literature and it also enables using standard approaches to obtain firm specific predictions of inefficiency (see Greene 1990, for details about the normal-gamma model).

Thus to identify the two parameters of the gamma distribution we use that the third order central moment of the composed error term is equal to the negative of the third order central moment of the inefficiency term and also that the forth order cumulant of the composed error term equals the counterpart for the inefficiency term. For the gamma distribution we have:

$$\mu_{3,\epsilon} = \mu_{3,\nu} - \mu_{3,u} = -\mu_{3,u} = -4\alpha\beta^3,$$
$$\kappa_{4,\epsilon} = \kappa_{4,u} + \kappa_{4,\nu} = \kappa_{4,u} = 12\alpha\beta^4,$$

and the following expressions for the two unknown parameters:

$$\alpha = \frac{3\mu_{3,\epsilon}^4}{4\kappa_{4,\epsilon}^3}, \tag{5}$$

$$\beta = -\frac{\kappa_{4,\epsilon}}{3\mu_{3,\epsilon}}. \tag{6}$$

In the next section we demonstrate how to obtain necessary consistent moment estimators for obtaining consistent estimators of the parameters presented in this section.

4. Consistent Moment estimators

In this section we present consistent moment estimators as $N \to \infty$. To obtain consistent estimators of models with inefficiency terms with two-parameter distributions one need consistent estimators of the third and the forth central moments of the composed error term. For one-parameter distributed inefficiency terms it is enough with a consistent third central moment estimator.

For the forth central moment estimator we need the consistent estimators of the four first raw moments. We will demonstrated how to derive a consistent estimator of the forth raw moment. Consider the following average of the within-transformed composed error term:

$$
\begin{aligned}
&\frac{\sum_t \sum_i \hat{\ddot{\epsilon}}_{it}^4}{N(T-1)} = \frac{\sum_t \sum_i (\ddot{y}_{it} - \ddot{x}_{it}\hat{\beta})^4}{N(T-1)} = \frac{\sum_t \sum_i (\ddot{\epsilon}_{it} - \ddot{x}_{it}(\beta - \hat{\beta}))^4}{N(T-1)} \\
&= \frac{\sum_t \sum_i \ddot{\epsilon}_{it}^4 + 4\ddot{x}_{it}(\beta-\hat{\beta})\ddot{\epsilon}_{it}^3 + 6\left[\ddot{\epsilon}_{it}\ddot{x}_{it}(\beta-\hat{\beta})\right]^2 + 4\left[\ddot{x}_{it}(\beta-\hat{\beta})\right]^3 \ddot{\epsilon}_{it} + \left[\ddot{x}_{it}(\beta-\hat{\beta})\right]^4}{N(T-1)} \\
&= \frac{\sum_t}{T-1}\left[\frac{\sum_i \ddot{\epsilon}_{it}^4}{N} + O_p(N^{-1/2}) + O_p(N^{-1}) + O_p(N^{-3/2}) + O_p(N^{-2})\right] \xrightarrow{p} \frac{\sum_t E(\ddot{\epsilon}_{it}^4)}{T-1} \\
&= \frac{T}{T-1}E(\ddot{\epsilon}_{it}^4),
\end{aligned} \tag{7}
$$

where $\ddot{z}_{it} = z_{it} - \bar{x}_i$, $\bar{x}_i = \sum_t z_{it}/T$. For the forth equality Lemma 1 and Lemma 2 in Wikström (2013) is applied with suitable moment assumptions.[2]

The forth raw moment of the within-transformed composed error term can be written as follows:

$$
\begin{aligned}
&E(\ddot{\epsilon}_{it}^4) = E(\epsilon_{it} - \bar{\epsilon}_i)^4 = E(\epsilon_{it}^4 - 4\epsilon_{it}^3\bar{\epsilon}_i + 6\epsilon_{it}^2\bar{\epsilon}_i^2 - 4\epsilon_{it}\bar{\epsilon}_i^3 + \bar{\epsilon}_i^4) = \\
&E(\epsilon_{it}^4) - 4E(\epsilon_{it}^3\bar{\epsilon}_i) + 6E(\epsilon_{it}^2\bar{\epsilon}_i^2) - 4E(\epsilon_{it}\bar{\epsilon}_i^3) + E(\bar{\epsilon}_i^4)
\end{aligned} \tag{8}
$$

where

$$
E(\epsilon_{ut}^3\bar{\epsilon}_i) = E\left(\epsilon_{ut}^3\frac{\sum_t \epsilon_{it}}{T}\right) = E\left(\frac{\epsilon_{it}^4}{T} + \frac{\sum_{t\neq u}\epsilon_{it}^3\epsilon_{iu}}{T}\right) = \frac{E(\epsilon_{it}^4)}{T}, \tag{9}
$$

$$
\begin{aligned}
&E(\epsilon_{iu}^2\bar{\epsilon}_i^2) = E\left(\epsilon_{iu}^2\frac{\sum_t \epsilon_{it}^2 + \sum_t\sum_{s\neq t}\epsilon_{it}\epsilon_{is}}{T^2}\right) = \\
&E\left(\frac{\epsilon_{it}^4 + \sum_{t\neq u}\epsilon_{it}^2\epsilon_{iu}^2 + \epsilon_{iu}^2\sum_t\sum_{s\neq t}\epsilon_{it}\epsilon_{is}}{T^2}\right) = \frac{E(\epsilon_{it}^4) + \sum_{t\neq u}E(\epsilon_{it}^2)E(\epsilon_{iu}^2)}{T^2} = \\
&\frac{E(\epsilon_{it}^4)}{T^2} + \frac{(T-1)\sigma_\epsilon^4}{T^2},
\end{aligned} \tag{10}
$$

[2] Where p denotes 'weak convergence in probability'.

$$E(\epsilon_{iu}\bar{\epsilon}_i^3) = E\left(\epsilon_{iu}\frac{\left(\sum_t \epsilon_{it}\right)^3}{T^3}\right) +$$

$$E\left(\epsilon_{iu}\frac{\left(\sum_t \epsilon_{it}^3 + 3\sum_t \sum_{s\neq t} \epsilon_{it}^2\epsilon_{is} + \sum_t \sum_{s\neq t} \sum_{r\neq t,s} \epsilon_{it}\epsilon_{is}\epsilon_{ir}\right)}{T^3}\right) =$$

$$E\left(\frac{\epsilon_{it}^4 + \sum_{t\neq u} \epsilon_{it}^3\epsilon_{iu} + 3\sum_{s\neq t} \epsilon_{it}^3\epsilon_{is} + 3\sum_{t\neq u}\left(\epsilon_{it}^2\epsilon_{iu}^2 + \sum_{s\neq t,u} \epsilon_{it}^2\epsilon_{is}\epsilon_{iu}\right)}{T^3}\right) +$$

$$E\left(\frac{\epsilon_{iu}\sum_t \sum_{s\neq t} \sum_{r\neq t,s} \epsilon_{it}\epsilon_{is}\epsilon_{ir}}{T^3}\right) = \frac{E\left(\epsilon_{it}^4\right)}{T^3} + 3\frac{\sum_{t\neq u} E\left(\epsilon_{it}^2\epsilon_{iu}^2\right)}{T^3} =$$

$$\frac{E\left(\epsilon_{it}^4\right)}{T^3} + 3\frac{\sum_{t\neq u} E(\epsilon_{it}^2)E(\epsilon_{iu}^2)}{T^3} = \frac{E\left(\epsilon_{it}^4\right)}{T^3} + 3\frac{(T-1)\sigma_\epsilon^4}{T^3},$$

$$E(\bar{\epsilon}_i^4) = \frac{\sum_t E(\epsilon_{it}^4)}{T^4} + 4\frac{\sum_t \sum_{s\neq t} E(\epsilon_{it}^3\epsilon_{is})}{T^4} + 3\frac{\sum_t \sum_{s\neq t} E(\epsilon_{it}^2\epsilon_{is}^2)}{T^4} + \tag{11}$$

$$6\frac{\sum_t \sum_{s\neq t} \sum_{r\neq t,s} E(\epsilon_{it}^2\epsilon_{is}\epsilon_{ir})}{T^4} + \frac{\sum_t \sum_{s\neq t} \sum_{r\neq t,s} \sum_{q\neq t,s,r} E(\epsilon_{it}\epsilon_{is}\epsilon_{ir}\epsilon_{iq})}{T^4} =$$

$$\frac{\sum_t E(\epsilon_{it}^4)}{T^4} + 3\frac{\sum_t \sum_{s\neq t} \sigma_{\epsilon_t}^2\sigma_{\epsilon_s}^2}{T^4} = \frac{E(\epsilon_{it}^4)}{T^3} + \frac{3(T-1)\sigma_\epsilon^4}{T^3}$$

For the expression in (11) the multinomial theorem is used for the first equality, mean-independence assumptions, $E(\epsilon_{it}|\epsilon_{is}) = E(\epsilon_{it})$ for all $t \neq s$, for the second equality and constant raw moments up to order four for the third equality.

Substitute equations (9)-(11) into (8) and simplify to obtain:

$$E(\ddot{\epsilon}_{it}^4) = \frac{T-1}{T^3}\left((T^2 - 3T + 3)E(\epsilon_{it}^4) + (6T-9)\sigma_\epsilon^4\right)$$

and furthermore substitute this expression into (7) and we obtain:

$$\frac{\sum_t \sum_i \hat{\ddot{\epsilon}}_{it}^4}{N(T-1)} \xrightarrow{p} \frac{1}{T^2}\left((T^2 - 3T + 3)E(\epsilon_{it}^4) + (6T-9)\sigma_\epsilon^4\right).$$

This gives us the following expression:

$$\frac{T^2\frac{\sum_t \sum_i \hat{\ddot{\epsilon}}_{it}^4}{N(T-1)} - (6T-9)\sigma_\epsilon^4}{(T^2 - 3T + 3)} \xrightarrow{p} E(\epsilon_{it}^4)$$

for an infeasible estimator of the forth raw moment of the composed error. This easily becomes a feasible estimator by replacing σ_ϵ^4 with a consistent estimator.

Similarily one can find the following consistent raw moment estimators:

$$\hat{E}(\epsilon_{it}) = 0, \tag{12}$$

$$\hat{E}(\epsilon_{it}^2) = \frac{\sum_i \sum_t \hat{\hat{\epsilon}}_{it}^2}{N(T-1)} \xrightarrow{p} E(\epsilon_{it}^2) = \sigma_\epsilon^2, \tag{13}$$

$$\hat{E}(\epsilon_{it}^3) = \frac{\sum_i \sum_t \hat{\hat{\epsilon}}_{it}^3}{N(T-1)} \frac{T}{(T-2)} \xrightarrow{p} E(\epsilon_{it}^3), \tag{14}$$

$$\hat{E}(\epsilon_{it}^4) = \frac{\sum_i \sum_t \hat{\hat{\epsilon}}_{it}^4}{N(T-1)} \frac{T^2}{(T^2-3T+3)} - \frac{3(2T-3)}{T^2-3T+3} \hat{E}(\epsilon_{it}^2)^2 \xrightarrow{p} E(\epsilon_{it}^4). \tag{15}$$

Now it is only left to find consistent estimators of the third central moment and the forth cumulant. This is straightforward by using the estimators in (12)-(15) and the following generic expressions of the third and fourth central moments and of the forth cumulant:

$$\mu_{3,\epsilon} = E(\epsilon_{it}^3) - 3E(\epsilon_{it})E(\epsilon_{it}^2) + 2E(\epsilon_{it})^3$$
$$\mu_{4,\epsilon} = E(\epsilon_{it}^4) - 4E(\epsilon_{it})E(\epsilon_{it}^3) + 6E(\epsilon_{it})^2 E(\epsilon_{it}^2) - 3E(\epsilon_{it})^4$$
$$\kappa_{4,\epsilon} = \mu_{4,\epsilon} - 3\mu_{2,\epsilon}^2 = \mu_{4,\epsilon} - 3E(\epsilon_{it}^2)^2$$

4.1. **Remark about asymptotic normality.** We are working with sample moments and asymptotic normality should be derivable under appropriate moment conditions. However, this will be a very tedious exercise. Wikström (2013) derives asymptotic normality for consistent estimators of raw moments up to order two and for consistent covariance matrices in this case, estimators up to order four are required. To derive a consistent estimator of covariance matrix of the gamma-normal estimators it will require estimators of raw moments up to order eight as well as covariances between all of these estimators. Therefore we conjecture asymptotic normality and propose bootstrapping for inference. For example assuming independent cross-sectional observations and make replications from $\{x_{it}, y_{it}\}_{t=1}^T$, $i = 1, 2, \ldots, N$.

5. Conclusions

In this paper we have demonstrated how to derive consistent estimators of the so called "true fixed effects" model based on method of moments. We think this offers more flexibility than provided by the likelihood methodology of Chen et al. (2014). We exemplify this by deriving consistent estimators of both the half-normal-normal and the gamma-normal stochastic frontier models.

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