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EDITORIAL

The 20th MAVI conference was organized by the Dalarna University. The conference took place from September 29th to October 1st 2014. There were 29 participants from nine different countries: Austria, Canada, Denmark, Finland, Germany, Israel, Italy, Switzerland and Sweden. Most of the papers had a research focus on teachers (Ándrà & Liljedahl, Bosse & Törner, Girnat, Larsen, Palmer, Pehkonen & Portaankorva-Koivisto, Tirosch, Tsamir, Levenson, Tabach & Barkai, Teledahl & Sumpter). Here we find paper about e.g. teacher change through participation in a research project and teachers’ curricular beliefs of teaching analytic geometry and linear algebra. We had also papers studying upper secondary school students and their reasoning (Jäder, Sidenvall & Sumpter, Olsson) and one paper looking at parents’ utility-oriented beliefs about mathematics (Albersmann). One paper studied MOOC (Massive open online courses) and the beliefs of MOOC producers. The diversity of papers indicates that the research in affect is still growing.

This proceeding contains the papers that were accepted to be presented at the conference. The papers are peer-reviewed, and the improvements from the first submission to this printed version are made based on feedback both from the reviewers, the editor and from the presentation. Every author is responsible for his/her own text.

The MAVI conference was initiated 1995 by Erkki Pehkonen and Günter Törner as a Finnish-German cooperative effort. It soon expanded to become an international affair, and since then it has been an important meeting point for researchers interested in affect and mathematics education. It provides opportunities to discuss research for a longer period of time; it is not uncommon for a discussion to last for days. It is the meeting between junior and senior researchers that makes this conference unique and valuable. There are now many senior researchers who have “grown up” with MAVI.

Normally at MAVI, they are no keynote speakers, which gives each participant equal status: all presentations are given the same amount of time for presentation and discussion. This year, in order to celebrate the 20th conference, three speakers were invited to give a presentation about the context of research in affect: where are we now (and why) and where are we going? The three speakers were Günter Törner, (a brilliant stand-in for Bettina Rösken-Winter who unfortunately had to cancel last minute), Peter Liljedahl, and Jeppe Skott. Each of these three gave a different account about what lies in the future of research in affect generating lot of discussions. As a research community, we need these types of debates to have the prospect of growing
as well as to avoid “scientific in-breeding”. This is what makes MAVI a precious place.

I would like to take the opportunity to thank the mathematics education group at Dalarna University for all the help they have provided in order to make MAVI 20 a wonderful experience.

The conference was funded by The Bank of Sweden Tercentenary Foundation.

Lovisa Sumpter
BELIEFS – NO LONGER A HIDDEN VARIABLE, BUT IN SPITE OF THIS: WHERE ARE WE NOW AND WHERE ARE WE GOING?

Günter Törner
University of Duisburg-Essen, Germany

This paper is based on a talk, given at MAVI-20, resulting from the intention of providing a survey of a multi-faceted landscape of research. Metaphorically speaking, it is like walking along a geological excursion path through older and younger regions.

INTRODUCTION – THE MAVI STORY

It is my personal conviction that in mathematics education, memories about developments are short-living, i.e. at most 30 years. As a senior, I have the privilege of looking back on sixty years of development in the field of mathematics education in Germany and I often wonder that my younger colleagues are not aware of the main historical facts and steps. The same is true for the history of beliefs and so please accept my attempt of throwing some light onto the progress made already in older times.

As many within my audience may know, the MAVI-story started in October 1994 with a German-Finnish conference in Duisburg. The abstracts of the talks are still available ([MAVI 1965]). MAVI is still alive, maybe even more alive than at the beginning. Twenty years ago there were exactly two professors; today we are many more. What are the ‘educational’ characteristics of the MAVI-network? What is the explanation for our success?

- **Content:** There is vivid and never ending research on beliefs, presented not only at the meetings of the PME, resp. PME-NA annual conference, but also in Europe. The more there are answers, the more we have questions about beliefs, no longer only about teachers and students.
- **Networking:** At the beginning there were scientists from two countries: Finland and Germany; by now, MAVI has become an international network which is more than indispensable.
- **Audience:** MAVI especially addresses young researchers in order to encourage them to dare to present their results. Today, young researchers who have started their careers with the support of MAVI are established researchers.

And finally:

- **Atmosphere:** All conferences take place in a very friendly atmosphere.
DEFINITIONS OF BELIEFS – NOT SATISFACTORILY SOLVED

This paper deals with beliefs, just as many other papers in different research journals do. Thus we should start with some definitions in order to clarify our intention – and this is something, which is usually not done in most of the other papers. Here we encounter a central problem: Although many authors work with the construct of ‘beliefs’, there is no standardized definition, which is accepted worldwide, as will be explained later on (see [Tö 2002]).

How to define beliefs?

According to Calderhead ([Ca 1996]) and Pajares ([Pa 1992]), terms such as ‘values’, ‘attitudes’, ‘judgments’, ‘opinions’, ‘ideologies’, ‘perceptions’, ‘convictions’, ‘conceptions’, ‘conceptual systems’, ‘preconceptions’, ‘dispositions’, ‘implicit theories’, ‘personal theories’ and ‘perspectives’ are often used almost interchangeably and sometimes it is difficult to identify their distinguishing features. Nevertheless, the openness and the vagueness of the definition explain the success of the terminology (see [Bo 1986], p. 2). Hence, my hypothesis is: Whatever the notion of the term ‘belief’, it may solve our problem.

However, there is also an opposite side: To some authors the term ‘belief’ appears too worn and they have decided ‘to invent’ a new terminology. Schoenfeld for example has decided to base his research on a more inclusive terminology, namely that of ‘orientations’. We cite ([Sch 2010], p. 29):

“I use the term orientations as an inclusive term, encompassing a group of related terms such as dispositions, beliefs, values, tastes, and preferences.”

How people see things (their ‘world views’, their attitudes and their beliefs regarding people and objects they interact with) shapes the very way they interpret them and react to them. In terms of socio-cognitive mechanisms, people’s orientations influence what they perceive in various situations and how they frame those situations for themselves. They shape the prioritization of goals, which are established for dealing with those situations and the prioritization of the knowledge that is used in the service of those goals.

Implicit definitions of beliefs

First of all, we will introduce the terminology ‘belief object’, which we have adopted from the construct ‘attitude object’ in attitude theory (see [EC 1992]) in order to describe the primary context to which the belief in question is related.

Belief objects may be small, e.g. some formula or some term, they may also be large, e.g. subdomains of mathematics or mathematics at school, at university or mathematics in general. The term ‘belief’ might interchangeably be used with the terms ‘ideology’ or ‘philosophy about mathematics’. Often, epistemological colorings may function as
beliefs. Some belief objects address social contexts and frameworks, e.g. the learning or teaching of mathematics. Self-conceptions about oneself as a learner of mathematics are also beliefs of the object self. It seems as if there is hardly anything to which there is no belief attached.

To the author it appears that functional characterizations of beliefs are much more important than any restricted, open or inclusive definition. Abelson ([Ab 1986]) refers to the ‘nature of beliefs’ and in particular, it is the nature of beliefs we are often faced with. In literature it was already established in 1973 that changes of beliefs are not easy to accomplished, cf. ([Th 1992], p. 139):

“Taken together, the results of Collier (1972) and Shirk (1973) suggest that the conceptions of prospective teachers are not easily altered, and that one should not expect noteworthy changes to come about over the period of a single training course.”

Thus we may find numerous (and different) metaphorical descriptions in literature. We will list some examples and refer to Katrin Rolka’s PhD thesis, where this aspect is further examined ([Ro 2006]).

- Beliefs are ... subjective explanations of the world, eventually only local, subjective or personal theories in the context of mathematics education (see Köller; Baumert; Neubrand [KBN 2000]).

In an early paper ([Sch 1985]) Schoenfeld used the terminology ‘worldview’, which is a perfect match to the German term ‘Weltbild’. It is obvious that worldviews cannot be discarded at once. On the contrary, beliefs per se are more resistant and stiff.

- Beliefs are . . . reductionistic views, think of barroom clichés. They reduce a complex world to a few characteristics, e.g. mathematics – the world of formulas.

Insofar, beliefs operate as limiting glasses or selective filters, which simplify a complex surrounding. They are partly appropriate; however, in places they are rough and neglecting. They represent correct cores; however, they don’t cope with the full reality. Again, it is convincing that individuals are willing to exchange their simplified views for complex perceptions.

- Beliefs are . . . like ‘Spaghettibundels’ (belief bundles). Thus, their contents and information cannot easily be stripped of their neighboring expectations.

Beliefs seldom occur as isolated or single objects, so that the exchange of isolated beliefs is nearly impossible. Therefore research also focuses on belief systems (e.g. see [TöP 1996]).

Beliefs are . . . like possessions (Abelson [Ab 1986], Rolka [Ro 2006]), which have been acquired or consolidated at length. This may explain why an individual is not willing to secede.

The metaphor ‘possession’ seems to be particularly central. We cite Schommer-Aikins ([SA 2004], p. 22):
“Beliefs [are] like possessions. They are like old clothes; once acquired and worn for a while, they become comfortable. It does not make any difference if the clothes are out of style or ragged. Letting go is painful and new clothes require adjustment.”

During the 80s there was a not very fertile discussion on how to distinguish between knowledge and beliefs (see [Ab 1979]). In mathematics the situation is quite easy: Knowledge is information certified by proofs; however, the world of beliefs and knowledge is much more complicated and only in rare cases can we apply this (theoretical) distinction. If we believe, then we may accept information as true which could be refuted by different means.

- Beliefs are . . . information which is ‘taken for granted’ as knowledge. Thus beliefs often serve as space fillers.

Some years ago we investigated the mental representations of exponential functions (see [BTö 2002]), in other words: We tried to draw a mind map of this subject. We had to realize that in complex networks, beliefs often serve as stabilizing knots when explicit (or proved) knowledge is not available.

We know from psychology that some individuals will not be satisfied by shaky world views. Insofar it is understandable that these people will refer to beliefs, which strengthen their own position. Therefore the development of identity and the cosmos of beliefs are intensively linked.

- Beliefs serve as . . . self-amplifiers or self-certifiers.

This again emphasizes that belief structures seem to be very stable.

**WHAT ARE THE MAIN MESSAGES? WHAT ARE ‘SOLID FINDINGS’ IN THE RESEARCH DOMAIN OF BELIEFS?**

The Committee for Education of the learned society of the European Mathematical Society (EMS), which is chaired by the author, edits a series of articles (for mathematicians!) in its newsletter. The title of the series reads “Solid findings”. The article ([EMS 2013]) written by the author deals with the topic of beliefs. What are today’s ‘solid findings’ with respect to beliefs? What do we know better now than 40 years ago?

**We are surrounded by beliefs... We are highly influenced by beliefs**

Now more than ever before, I know that we are surrounded by beliefs. In order to phrase it according to the title of a famous book by Lakoff ([LJ 1980]): We state beliefs we live by.

We should note that beliefs played a dominant role in the failure of the curriculum reform in the United States during the the 80s. It was Marta Frank who studied in her PhD (1985) the (potential) incompatibility between problem solving and teachers’ and students’ inherent beliefs (see [Fr 1985], [Fr 1988], [Fr 1990]) and also Schoenfeld
who wrote papers on problem solving, e.g. [Sch 1994]), at that time. What happened in the US in the midth-1980s should have happened again many times; however, it has never been as explicitly stated as it was stated during the 80s.

Tobin and LaMaster (1992) ([TLm 1992]) phrased these insights as follows:

“However, what became apparent was that teachers implemented the curriculum in accordance with their own knowledge and beliefs and did not necessarily do what curriculum designers envisioned. Several studies . . . indicated that teachers do what they do in classrooms because of their beliefs about what should be done and how students learn.” (p. 115)

It is very complicated to avoid beliefs. We have to be aware of that! It is the author’s opinion that beliefs do not occur isolated or randomly, but that we are often faced with systems of belief. What has recently become more and more evident in mathematics education is the fact that beliefs turn out to serve as important modules within larger theoretical contexts (see Section 6).

**Beliefs are a matter of ideology and philosophy**

An important success of the MAVI-network is the Leder-Pehkonen-Törner-book ([LPT 2002]) which was published in 2002; however, we had already started to work on it in the late 90s. The title underlines that at that time beliefs were hidden variables.

Thus beliefs often influence our epistemology. Beliefs serve as linkages between mathematics and its didactics. Again I should cite René Thom’s important quotation ([Tho 1973]): “In fact, whether one wishes it or not, all mathematical pedagogy even if scarcely coherent, rests on a philosophy of mathematics.”

Paul Ernest (e.g. [Er 1991a], [Er 1991b]) has worked on this linkage in particular.

**Beliefs as a source of explanation in various situations – Beliefs: A wild card?**

Of course, many times it has been claimed that the behavior of teachers, students and professors is rooted in their beliefs. Until today the correlation between action, activities and beliefs is neither understood in general nor in detail; however, we are not able to prove the contrary. Therefore, we state and assume an influence of beliefs.

There is an old question, which hasn’t finally been answered: What is the relationship between students’ beliefs about mathematics and achievement (see e.g. [Kl 1991] or [TW 1983] and many further papers)? A similar question has been asked by [DT 1989]: Do teachers’ beliefs and qualities influence students’ beliefs about mathematics?

Once again, the author is ready to accept some influence of beliefs, but there is no direct implication (see the arguments in Schoenfeld ([Sch 2002]).

**Beliefs are respected by mathematics education researchers everywhere**

In 2002 the author in the Kluwer book described beliefs as hidden variables. About ten years later, we were convinced to publish a paper declaring that beliefs are no longer
hidden variables ([GRT 2008]). Meanwhile hundreds of papers have been published and there is almost no large conference not offering a session on beliefs. Furthermore we do now have colleagues and specialists working in the field of beliefs in countries where beliefs weren’t one of the main research topics twenty years ago.

WHAT CAN WE LEARN FROM HISTORY? – THE HERITAGE OF BELIEFS

In mathematics it is easy to mark progress by examining which problems have been solved in the last ten, twenty or fifty years and which remain unsolved. Telling a similar story in mathematics education is much more complicated.

The attitude heritage of beliefs

MAVI should be aware of this heritage. Already in the 1940s ([Bi 1944]) researchers investigated the question of whether attitudes may have influence on the reception of mathematics and its assessment. The researchers regarded this issue as a topic of attitude. Essential for an attitude at that time were its affective side and implications.

Articles on this subject provide an affirmative answer, which may be astonishing for an expert. In the 1950s it was learnt that in addition to the rather obvious affective side, subjective understandings of mathematics and mathematics teaching influence the reception of the content even more and may thus lead to an unsatisfactory assessment of mathematical topics ([Du 1951], [BH 1954], [MF 1954], [Tu 1957], [Fed 1958]). Even more so, they may affect engagement structures, as they are called today by Goldin et al ([G etal 2011]).

Gradually, three further aspects have emerged: The question of how to measure attitudes ([Du 1954], [DuB 1968]), the extension of the problem from local aspects of different subdomains of mathematics to mathematics as a larger field ([PN 1959]), and finally the decisive problem of how to change attitudes ([Du 1962]).

In none of these older publications is the term ‘belief’ even mentioned. The authors’ grounded theories are the convictions that attitudes govern behavior and therefore attitudes are a decisive variable of success and failure in mathematics. I believe that these old articles contain valuable insights and hidden treasures that need to be found. Some of the papers contain questionnaires and the authors tried to establish attitude scales.

Today we would no longer match the observations with attitude theory, but rather speak of epistemological beliefs.

Beliefs in the context of sociological aspects within teaching and learning

It seems to me that around 1960 the term ‘belief’ came in use in other areas, particularly in educational communities. In particular, I assume that at least the following publications made the term ‘belief’ socially acceptable ([H etal 1966]), [H etal 1968], [Ro 1968], [Gr 1971], [Ab 1979]).

Rokeach (1968) offered the following descriptions:

1. Belief: ‘... any simple proposition, conscious or unconscious, inferred from what a
person says or does, capable of being preceded by the phrase ’I believe that . . .’” (p. 113).

2. **Attitude:** ‘… a relatively enduring organization of beliefs around an object or situation predisposing one to respond in some preferential manner’ (p. 112).

3. **Value:** ‘… a type of belief, centrally located within one’s total belief system, about how one ought or ought not to behave, or about some end-state of existence worth or not worth attaining’ (p. 124).

4. **Opinion:** ‘… a verbal expression of some belief, attitude, or value’ (p. 125).

It is obvious that the distinction between these four terminological constructs is not easy, it is nearly virtual. Some years later, attitude theory emancipated itself as an independent domain of research, thus the system of beliefs addressing educational problems remains. Fortunately, it has been observed that the beliefs of teachers are not less important than those of students (e.g. [Co 1972], [Fe 1978]).

**Beliefs and epistemology**

There is no space left to discuss the aspects and the correlation between beliefs and epistemology. Looking through older papers from the 1950s and the questionnaires which were used at that time in order to identify attitudes, one soon discovers that many questions around beliefs are epistemological in nature.

But how does epistemology influence the daily classroom? Epistemology is an interesting subject for theoretical research, but how do epistemological beliefs interact with the teachers’ activities?

**HOMEWORK TO BE DONE – WITH RESPECT TO BELIEFS**

**Beliefs are still a fuzzy construct**

However, there are still some shadows we should not overlook. There is homework to be done.

**Belief papers often miss the reference to grounded theories**

To be honest, there is no uniquely determined theory and I accept various anchoring theories. Seldom do authors of belief papers make their anchoring in a specific theory transparent.

There are authors or handbook articles to whom/which researchers might refer. By the way, the appearance of handbook articles also emphasizes that beliefs become meanwhile well-established (see e.g. [Th 1992], [Ca 1996], [R 1996], [Ph 2007], [MaS 2008] and so forth).

**The terminology is still not standardized**
More than ten years ago the author compiled some elements of a theory in his book ([Tö 2002]), an attempt that might not have been perfect. However, it introduced and proposed a very helpful terminology borrowed from attitude theory: The term

- belief object

which seems quite helpful to me. We often speak about beliefs of person, but beliefs about what? The students’ ideas? The teachers’ visions? The reality in the classrooms? The reality stated by the answers? Or by the text in a homework? We have to encircle the topic, to narrow down the problem and thus to identify the belief objects which might be conceived in a different way in a different paper.

Emotions may be a feature of beliefs, but it is not necessary to associate some feelings with an arbitrary belief. The answer to the multiplication task 7×8 may be accompanied with affections, since the unknown answer may be conceived as a difficult problem and thus be hated by the person who is supposed to solve it. And in general there is indeed hate towards arithmetic. For us however, 7 times 8 is just 56, but nothing difficult.

Thus emotions may be attached to beliefs, but they are not universal.

**The open question: Beliefs and behavior**

Historically belief research history one hoped that beliefs were keys in order to explain and understand behavior. It was Triandis ([Tr 1971]) who has postulated the three facets, A = affective, B = behavioral and C = cognitive sides of attitudes (and beliefs). Many papers on beliefs have claimed and still claim that beliefs have implicit implications for behavioral characteristics and so did Schraw and Olafson in about 2002. Their paper ([SchrO 2002]) induced an intensive discussion in the literature; once again it has been demonstrated that there is no easy explanation.

Today we know that goals and beliefs are closely related ([TRRS 2010]). And again: Goals are close to behavior. Maybe there are some indirect and not completely understood affiliations that let us presume that beliefs are responsible for activities.

**Partly understood: Change of beliefs resp. stability of beliefs?**

There is a long list of articles on how to change beliefs. I don’t want to comment on it. To put in in a nutshell: Are beliefs stable or not? In particular it was Peter Liljedahl (see [LiOB 2011]) who questioned the stability of beliefs during a MAVI meeting some years ago:

“I didn’t work in this field, but my proposal is to have a more careful view into the beliefs under discussion and then there might be some better explanations.”

**BELIEFS AS MODULES WITHIN LARGER THEORIES**

Metaphorically speaking we understand beliefs as atoms in a molecule structure. In the past we discussed and investigated just (plain) beliefs of teachers, students and other persons. Gradually we gained the insight that beliefs are often components within
larger structures. Here are some examples taken from my work and experience over the last four years.

**Beliefs and professional development**

When my PhD student Bettina Roesken started her research on professional development, she soon became aware of the fact that she had to handle beliefs in the context of professional development (see [Rö 2011], [Ma 2011], [Sw 2007] and [Ti 2008a]).

In particular, Timberley and her co-authors have pointed out that promoting professional development is developing beliefs. It is our intention to initialize changes in practice. Practice is based on teachers’ knowledge and ‘their beliefs about what is important to teach, how students learn and how to manage student behavior and meet external demands’ ([Ti 2008b]). The beliefs that might be problematic have to be challenged, reflected on and rethought. The efficacy of competing ideas which create dissonance has to be tested. This is a starting point for developing a more adequate understanding. Thus, beliefs play a key role in professional development; however, professional development consists of much more than of the analysis and change of beliefs.

**Beliefs and out-of-field teaching**

Two years ago my working group started approaching the question of out-of-field teaching. In many countries, teachers are active in mathematics classrooms without having any formal qualification for teaching the subject. My assistant Marc Bosse has intensively researched the phenomenon of this group of teachers and has soon detected that the identity development of these teachers could partly be mapped by their beliefs on teaching and receipting mathematics ([BTö 2013], [TT 2012]). At MAVI-18, our talk focused on the ambivalent role of beliefs and again pointed out the central role of teachers’ worldviews on mathematics.

**Beliefs and transition problems**

Within the life-long educational CV there are various incisions marked by a transition from one level of education to a further level of education.

For a long time the phases of transition have been a research topic since these changes are related to some kind of problem: From early childhood education to school education ([Ti 2003]), the transition from school to university (e.g. [GHRV 1998], [Gu 2008], [CL 2009]), the so-called secondary-tertiary transition, the transition from a prospective teacher to a novice-teacher, from a ‘learner’ to a ‘teacher’ (e.g. [J et al 2000]). Dozens of further papers could be cited. More than half of the papers report on
conflicting beliefs at these two different stages; other papers, which do not directly refer to this terminology, describe constructs which can be interpreted as beliefs.

**Personal development of novice teachers and the role of beliefs**

We want to bring this discussion to an end by giving a fourth example: A few days ago I stumbled across a paper by Skott ([Sk 2001], see also [LC 2013] and [Ra 1997]) and again ‘images’ – more or less beliefs – were an important variable for understanding the teaching activities of novice teachers.

**Further candidates of theories involving beliefs**

Just take the book of B. Sriraman and L. English ([SE 2010]) and browse through its pages, you will soon bump into articles which implicitly or even explicitly refer to beliefs, e.g. [TRRS 2010] in which teachers’ actions in classrooms are analyzed.

Further interesting candidates with belief modules would be didactical contracts, values, norms and many more.

**CONCLUSIONS**

Looking back at this compact article it was the author’s intention to shed some new light on the ancestries of beliefs which are – because of the non-standardized terminologies – partly ignored by the actual research, since beliefs at that time have often been called attitudes. The author believes that it is worthwhile to look for older insights and observations since there might be some buried treasures to be found.

Finally, belief researchers should also think ‘big’: They should think of recognized modules in various theories of mathematics education to which the belief theory could be applied.

**References**


Törner


EMOTIONS AS AN ORIENTING EXPERIENCE

Peter Liljedahl
Simon Fraser University

In this article I present the results of a research project embedded within a larger narrative about the tensions between the participationist and acquisitionist theoretical framework. The research explores data collected from 38 prospective elementary teachers after an intensely negative emotional experience preparing to play a game called Around the World. From these data emerges a picture of these prospective teachers being deeply affected by this experience. To try to understand these changes I use Activity Theory as formulated by Leont’ev. This theory, based on Vygotsky’s cultural historical theory, looks at the relationship between motives, activity, and emotions. Using this theory I argue both theoretically and empirically that what has actually changed for these prospective teachers are their motives. More specifically, the hierarchy of their motives. The results are one of the few contributions in mathematics education that anchors emotions in a theoretical framework and links them to other constructs in the affective domain.

FOREWORD

In the fall of 2009, Magnus Österholm organized a meeting in Umeå, Sweden called the Workshop on Mathematical Beliefs (WoMB). This workshop was built around the plenary addresses of four main people: Markku Hannula, Erkki Pehkonen, Jeppe Skott, and myself. Pehkonen gave an “Overview of Empirical Results about Beliefs”, Hannula presented “A Reflection of Belief-Research Including Some Unresolved Issues”, I gave a talk on the “Stability of Beliefs”, and Skott gave a presentation, innocently called, “Individual and Social Aspects of Beliefs”.

Skott’s talk, however, was anything but innocent. In it he argued for a theoretical framework, called Patterns of Participation, which had evolved out of the frameworks of Lev Vygotsky, Etien Wenger, and Anna Sfard. He further argued that, not only does this framework better account for teachers’ actions than beliefs research, but also that beliefs, as a construct, do not even exist. Understandably, this created quite a stir.

After these plenary addresses the four of us, plus Magnus, spent two days in a room arguing the merits of our varying and conflicting frameworks. In many ways, this was futile. Skott’s framework, evolving from the participationist traditions of Vygotsky was, by default, incommensurable with the acquisitionist roots of beliefs research. I was not convinced. There was too rich a basis of research on beliefs to, so easily, discount the validity of this construct. At the same time, there were merits in Skott’s arguments. I felt that there must be a bridge.
INTRODUCTION

In the spring of 2010 a series of events conspired to put me on a path towards searching for this bridge. It began with a visit from Kim Beswick to my EDUC 475. EDUC 475 is the mathematics method course for prospective elementary school teachers. Each section of the course usually has 30-35 students, 90%-95% of whom are female. On the particular day that Kim visited we were discussing basic operations on single digit numbers – addition, subtraction, multiplication, and division. The goal of the lesson was to get the students to experience methods of teaching these operations other than memorization and rapid recall, which is the only method familiar to many of them.

Although the lesson has this goal, this only defined the general direction I wanted to go in. During the actual lesson I draw on a large repertoire of activities and discussion points that tumble out in a, more or less, improvised order. This allows me to more effectively respond to my perceived needs of the specific group of students at that specific time.

As it was, many of prospective teachers I was teaching the day Kim visited, although seeing the merit to the many alternative methods I was modelling, were still not ready to abandon the ‘drill’ method of teaching fluency of the basic facts. Many had mentioned at the beginning of this lesson, as well as in the previous lesson, that they had regularly used The Mad Minute during their practicum. This was problematic to me. The Mad Minute is a test, usually given once a week, where students are challenged to answer 30 questions in one minute. Their scores on these tests are often recorded in some public fashion and the top achieving students are rewarded for their achievements. The possible negative consequences of this method are many, yet it continues to be practiced for its efficiency, simplicity, and tradition ... and parents like it.

To emphasize the potentially negative consequences of this method I did something I had never done before. After the pre-service teachers returned from a break I gathered them around me. I told them that we were going to do a basic facts activity. The way this activity would work is that I would point at one of them and ask them a basic multiplication question (3 × 4, 6 × 8, etc.) and they would have two seconds to respond. If they responded correctly in that time they would be allowed to sit down. If they failed to give response, or their response was incorrect, they would remain standing and I would come back to them after I had gone all the way around the class. This would continue until all the students were sitting.

This game, as it is referred to by practicing teachers, is called Around the World, and is often used, in conjunction with The Mad Minute, as a way for students to practice their basic facts. Unfortunately, it has the same sort of public shaming qualities that the Mad Minute does.

The pre-service teachers gathered around me were, as a group, visibly uneasy. There were a few who seemed excited at the prospect of playing a ‘game’ and the thrill of
competition. But the vast majority were horrified at what was about to happen. When the tension had built to a crescendo I pointed at the first prospective teacher and, instead of asking a basic multiplication question, asked, “How are you feeling right now?” And then, to the whole group, “How are all of you feeling right now?”

The relief in the room was tremendous, and the ensuing conversation was beyond anything I had expected. The experience of almost having to play Around the World was transformative for these soon-to-be teachers who talked about how they NOW understood how negative this game—and The Mad Minute—could be. For over an hour they talked about their past experiences, sharing the negative impact these types of ‘games’ had on them as learners. A few of them shared their positive experiences with these types of activates, but even then quickly acknowledged that their enjoyment was not worth the price of misery that the rest of the students had to pay. We discussed why parents liked these ‘games’ and ways, as future teachers, to deal with that. In the end they vowed, individually and as a group, that they would never do this to their future students.

After the class, in debriefing the activity with Kim, we both concluded the obvious – the prospective teachers had had a powerful emotional experience and that that experience had caused wide sweeping changes in their intended practice (Liljedahl, 2008). But, we also concluded that we currently had no theoretical framework to make sense of this experience.

In mathematics education research in general, and in affective research in particular, emotions remain a largely unresearched and not-well-understood construct. The little research that exists is either atheoretical with respect to the construct of emotions, or emotions are “sidelights rather than highlights of the studies” (McLeod, 1992, p. 582).

As such, we decided that we needed to recreate the phenomenon and to gather data on it.

**METHODOLOGY**

So, in the spring of 2012, working with a new group of 38 (35 female and 3 male) EDUC 475 students, I recreated the Around the World activity. As mentioned, EDUC 475 is an elementary mathematics methods of teaching course. It runs for 13 weeks and is comprised of 13 lessons – one each week. Each lesson is four hours long and is typically designed around a number of activities and resultant discussions. Between lessons, students are assigned readings and prompts to be responded to in a reflective journal. As with the previous class, many of the prospective teachers in the current class had acknowledged that they had used The Mad Minute or the Around the World activities during their practicum, either on their own initiative or at the urging of their sponsor teacher.

As such, in the fourth week of classes I once again ran the Around the World activity. This time, however, instead of immediately going into a discussion I did something
different. As the tension built to a crescendo I pointed at a student and asked her how she felt, and then I immediately asked her, and all her classmates, to sit down and write in their journal how they felt at that moment. The students wrote for 10-15 minutes. We then had a whole class discussion much as I had led for the class two years prior.

At the end of the class they were assigned a further journal prompt:

Discuss your experience in today’s class around the issue of multiplication. What did you feel when I sprung the “stand up and get ready to answer multiplication facts” activity? What sort of self-reflection did you go through? How do you feel now after we debriefed it?

Towards the end of the course, the students were given a further writing prompt potentially related to the Around the World activity and discussion.

Now that this course is almost over what is something that you will NEVER do in the teaching of mathematics? Why? What is something that you will ALWAYS do? Why?

Taken together, data consists of the relevant entries from the written journals of these 38 prospective teachers. These data were analysed using a constant comparative method (Glaser and Strauss, 1967) to emerge themes pertaining to their emotions and the effect of those emotions, both short term and long term.

RESULTS

For the most part, the game of Around the World created a very negative emotional experience for these prospective teachers.

Fear

Fear, in one of its many forms, was one of the most commonly expressed emotions immediately after the activity.

Misha Terrified! I can’t do mental math very quickly and I don’t like being the centre of attention when under scrutiny. The only thing I could think was “I’m going to be the last one standing”. I don’t want to look slow in front of my peers and teacher. Through my education career I sit in my seat praying not to be called on.

Allison Mortified. I don’t like to be wrong or feel embarrassed in front of my peers. It can be extremely difficult to get the answer right as I’m too busy thinking about me, or what they are saying to care about the problem. Eventually I feel I’d just guess to get it over with.

Anxiety

The other emotion frequently expressed is anxiety.

Beth Heart racing anxiety! The thought of being picked on and not knowing gives me the heebie jeebies, especially in a subject that is probably my weakest. Being that it is multiplication and is something that I probably would get right doesn’t really help shake the feeling you get when you
know that there is pressure to perform. [...] If I feel like this at 23 how would a kid feel?

Jocelyn  I am feeling really anxious and nervous. I am worried about being embarrassed about not being able to answer the multiplication question in front of the class and I am also really worried about being the last person standing.

Nervousness

Nalah  I felt nervous because I might not know the answer to the multiplication question he might ask. [...] While we were standing there waiting for Peter to ask, I was thinking back to grade two and three and how we played the game *Around the World*, and how nerve racking it was.

Defeated

Anne  It also reminded me of a time when my grade three teacher called me to the front of the class to answer a question. She knew I wouldn’t know it, but I had to do the walk of shame to the board only to admit to the whole class that I didn’t know the answer. I dreaded going to class. I just remember being in class and feeling defeated by math.

**TEACHER CHANGE**

These very negative emotions were not fleeting. Despite the fact that during the course we engaged in over 50 activities and discussions, and read over 400 pages mathematics education literature, six weeks after the *Around the World* activity, 24 of the 38 prospective teachers in the course chose to discuss this specific activity, and the emotions it triggered, when responding to the prompt about something they would never do in their teaching.

Misha  Something that I will NEVER do in the teaching of mathematics is put students on the spot and force them to answer questions. Like many other people, I have experienced embarrassment from being put on the spot and answering incorrectly. I understand how low it can make; a student feel and I don't want to be the one to make my class feel that way.

Sofia  In teaching math I will never use the Mad Minute to drill students on their multiplication tables. The costs to many students outweigh any benefits to a minority of students.

Jocelyn  Now that the course is over, I have discovered that I will never make my students do any sort of drill or mad minute that may deflate their confidence and cause them to want to avoid mathematics. I realize what effect 'mad minute' exercises had on me as a math student and when Peter simulated a mad minute situation, I felt terrified and extremely anxious. I would never want my students to feel that kind of panic and fear. As a teacher, I hope to foster a love for learning mathematics and want to create an environment whereby my students feel confident and safe.
Even those who were originally excited by the game talked about the negative emotions they say their classmates experience.

Alison  
To be honest, I was excited to play the game. But I can see how an activity like this could bring high levels of anxiety for students in a class that are insecure about their amount of knowledge or skills with respect to what is being quizzed on the spot. I did not feel panicked because I am confident that my multiplication skills are fine. […]Something I will NEVER do when I teach math will be multiplication drills. It traumatizes children that are not finding this activity successful, and it could give them a bad taste for math for the rest of their life.

Of the 14 who did not speak of the Around the World activity explicitly in their response, 12 made commitments that were tangential to some of the ideas that cascaded from subsequent discussions on the learning of basic facts in general, and assessment in particular.

Anne  
I will NEVER use assessment as a way to rank students.

Khaly  
I'm not afraid of mathematics any more, to learn or to teach. I also think that mathematics can actually be fun. I am excited to teach my new students (when I get my first class). Show them that math is not as scary as it seems.

Taken together, 36 out of the 38 prospective teachers, despite many having used it in their practicum, vowed to never use Around the World (or the Mad Minute) in their future practice as teachers. For them, their own experience with this activity had triggered very negative emotions, sometimes reminding them of similar activities and emotions from when they, themselves, were children. These emotions were not only enduring, but also instrumental in changing things in the prospective teachers’ practice.

But what exactly is it that has changed for these teachers? Given that they don’t actually have a classroom in which to enact these changes we cannot say that it is their practice that has changed. Perhaps it is their intended practice that has changed? But what is backstopping this intention? Intentionality is a reification of deeper constructs. The question is, what is the construct that grounds these intentions, that was deeply affected by the emotional experience that these teachers had when being placed in a position of having to play Around the World? To answer this we need to look more closely at emotions.

**EMOTIONS**

From an acquisitionist perspective, emotions are seen as the fleeting and unstable cousins of beliefs and attitudes (McLeod, 1992). They are either a reaction to an experience (McLeod, 1992) or a reaction to an interpretation of an experience (Mandler, 1984). Regardless, emotions are acknowledged to affect learning in general (Zan, Brown, Evans & Hannula, 2006) and cognitive processing in particular (Hannula, 2002). Over time, negative emotions can reify into more stable and
disassociated manifestations of fear, phobia, and hatred (DiMartino & Zan, 2012; Tobias, 1978), each of which will have an effect on actions (Hannula, 2002; Tobias, 1978).

That emotions exist, and that they simultaneously emerge from, and shape experience, is clear. That these emotions then regulate future actions is also clear. What is not clear, however, is how this happens. What psychological mechanisms link emotions to actions?

From an acquisitionist perspective I could say that it was their beliefs that were changing. The data certainly was implying this. However, to make this claim I would need to make the leap from emotions to beliefs, and acquisitionist perspectives did not allow for this. Not only were there no theories that linked these two constructs, within mathematic education there existed few, if any, viable theories of emotions. On the other hand, within the participationist perspective emotions were well grounded in theory.

This was my chance. If I could start with emotions in the participationist tradition and find, through these traditions, evidence of beliefs, I would have proven that it is possible to bridge these seemingly incommensurable stances.

EMOTIONS AND ACTIVITY

To begin with, in the participationist framework, emotions cannot be well examined in the abstract.

The variety of emotional phenomena and the complexity of their inter-relations and sources is well enough understood subjectively. However, as soon as psychology leaves the plane of phenomenology, then it seems that it is allowed to investigate only the most obvious states (Leont’ev, 1978, p. 168)

That is, emotions must always be considered in the context of the phenomena in which it occurred.

Consider a wolf in the wild. This wolf has a vital need to eat, and this need to eat drives him to hunt. These hunts result in him catching mice, rats, and rabbits. This then shifts the abstract need to eat into a concrete need to eat mice, rats, and rabbits. Then one day, he catches, for the first time, a duck. This, in turn, changes his need to include ducks in his menu of things he eats. And so on. Each time the wolf, through his hunt, encounters a new animal that he can eat, his needs change.

For Leont’ev (1978), such is the relationship between needs and activity. As humans, our vital needs, abstract and unrefined, drive our activity to satisfy these needs. These activities, grounded in phenomena, in turn gives an object to the needs.

The fact is that in the subject’s needy condition, the object that is capable of satisfying the need is not sharply delineated. Up to the time of its first satisfaction the need “does not know” its object; it must still be disclosed. (Leont’ev, 1978, p. 161).
This recursive relationship between needs and activity, each driving the other, expands and refines both the object of need, and the need itself, forming what Leont’ev (1978) refers to as **concrete-objective needs**. This, in turn, changes the subsequent action.

[...] it is understood that changing the concrete-objective contents of needs leads to a change in methods of their satisfaction as well. (Leont’ev, 1978, p. 162)

Sometimes, however, this recursive cycle shifts the need from the object to the activity itself, forming what Leont’ev (1978) calls an **objective-functional need**. These needs, such as the need to work, to be productive, or to be creative, for example, do not displace the original needs that spawned them, but come alongside them as additional new needs. It is important to note, however, that not all objective-functional needs come from a newly acquired focus on activity. Likewise, not all vital needs are based on objects. In a cultural historical framework, action oriented needs can be part of the milieu. For example, the need to be subservient, to strive for physical perfection, or to always clean, can be *a priori* embedded needs within a person’s specific cultural upbringing. Regardless, activity and “the satisfaction of the need” helps to delineate it.

For Leont’ev (1978), these delineated concrete-objective and objective-functional needs, in their ideal and reflected forms, are what he calls **motives**. And despite the fact that the language on needs and activity are shot through with willfulness and implied consciousness, our motives are not always known to us. Further, our activities are multi-motivational.

Such breaking down is the result of the fact that activity necessarily becomes multi-motivational, that is, it responds simultaneously to two or more motives. (Leont’ev, 1978, p. 169).

They organize themselves in hierarchies, and these hierarchies define, to a great extent, an individual’s personality.

A division of the function of sense formation and simple stimulation between motives of one and the same activity makes it possible to understand the principal relationships characterizing the motivational sphere of personality: the relationships of the hierarchy of motives. This hierarchy is not in the least constructed on a scale of their proximity to the vital (biological) needs in a way similar to that which Maslow, for example, imagines: The necessity for maintaining physiological homeostasis is the basis for the hierarchy; the motives for self-preservation are higher, next, confidence and prestige; finally, at the top of the hierarchy, motives of cognition and aesthetics. (Leont’ev, 1978, p. 170).

Finally, for Leont’ev (1978), **emotions** act as an internal signal within this relationship between our motives and the actions that work satisfy them. That is, despite the fact that motives could be unknown to an individual, when they are realized there is an emotional response that signals that success has been achieved.

Here we are speaking not about the reflection of those relationships but about a direct sensory reflection of them, about experiencing. Thus they appear as a result of
actualization of a motive (need), and before a rational evaluation by the subject of his activity. (Leont’ev, 1978, p. 166-167).

These emotions have the potential, then, to reorganize the hierarchical order of these motives.

For example, a businessman has a goal to earn more than $100,000 in his job as a sales manager. One day, his boss calls him into his office and tells him that he is receiving a raise and will now be earning $110,000 per year. The man is elated. Later on that same day he overhears that his colleague has also been given a raise and will now be earning $120,000. Suddenly, a feeling of dread comes over him. Reflecting on this negative emotional response the man comes to see that what he really wanted was to be the best sales manager in the company. Earning over $100,000 a year was not the primary goal. The primary goal was to be the best sales manager in the company. But that goal was hidden from the man.

In this example, the man’s surprising emotional reaction to hearing that his colleague was making more than him left him with an “emotional residue” (Leont’ev, 1978, p. 172) that moved him to engage with his hierarchical structure of motives, and to try to figure out what it is that is really driving him. In so doing, a motive that he was not previously aware of revealed itself. This results in a re-orientation of motives, which is tantamount to a re-orientation of his personality – all of which is triggered by his emotional response to an experience.

As such, in Leont’ev’s (1978) framework, emotions serve as the orienting mediator between action and motives, and between motives and personality. In short, an emotional response to a specific experience draws the attention of the individual to their motives and allows them to begin the cognitive process of re-orienting their motives’ hierarchy.

**EMOTIONS AROUND THE WORLD**

Leont’ev’s theory of emotions, motives, and personality, situated within the cultural-historical paradigm of individualized activity theory allowed me to look anew at the data from the prospective teachers playing *Around the World*.

All of these teachers wanted to be good teachers. This was one of their many motives. But they also wanted to please parents, have their students be good at basic multiplication facts, and to not make their students anxious or fearful, to name a few. These many goals were organized into hierarchies, unique to each prospective teacher. For the most part, these prospective teachers were not aware of many of their motives. Instead, they were fixated on their current goals of learning how to teach mathematics, getting good grades, and/or having their knowledge experience acknowledge. The “emotional residue” left from their experience playing *Around the World* helped them to see some of these motives. And it helped them to re-orient them.
In what follows I provide a brief case study on one of the prospective teachers—Tara—analysed through the lens of Leont’ev’s theory on motives, emotions, and personality (1978).

**Tara**

Immediately after the *Around the World* activity Tara wrote that she was feeling a little anxious.

Tara I’m feeling a little anxiety, because [she] did not want to look stupid if [she] got it wrong.

However, she also saw merit in this activity.

Tara As a teacher I see the value in this activity. Students must be ‘switched-on’ and engaged. It forces them to use their brains and everyone must participate. The likelihood of everyone getting the correct answer is unlikely so no one will feel bad if they don’t get to sit down. It also creates a competitive environment and opportunity for kids to shine.

The hierarchy of motives from Tara’s post-activity journal indicates that students being ‘switched-on’ is a primary motive for her as a teacher (see figure 1).

![Figure 1: Tara’s initial hierarchy of motives](image1)

![Figure 2: Tara’s final hierarchy of motives](image2)

Later that night, when responding to the journal prompt her motivations had changed somewhat.

Tara Today when Peter sprung the "stand up and get ready to answer multiplication facts" activity, my first feeling was fear and anxiety. I was worried I'd get the answer wrong and look stupid in front of the class. [...] This exercise made me think about my own classroom, how or whether I would use an activity like this. I think I would, that being said, would my students feel the same anxiety I did? Most likely they would, but I think after getting into the game they would enjoy the competition. The environment I plan to create in my class would assure them that I was not having them do this activity to humiliate them but to use mental calculation and practice their skills.
In that moment, Tara’s hierarchy of motives (see figure 2) now has concern for her student’s anxiety at the top and the motive to ‘switch-on’ students has dropped away. The few hours that she had to reflect on her experience with the activity and the emotional residue it left has caused a re-orientation of her motives.

And this residue endures. At the end of the course Tara continues to talk about her motive to not let students become anxious, although not quite as explicitly.

Tara I will never just stand at the front of a class and 'teach'. [...] Basically, I won't be afraid to teach outside the box ... the traditional box that I learned in and that we all know so well. I want to inspire my students. [...] I now know math doesn't have to suck ... the way it did for me in grade school.

CONCLUSION

So, what is it that changed for Tara and for the rest of the prospective teachers who ‘played the game’ of Around the World? According to Leont’ev’s Activity Theory, what has changed are their motives. More specifically, their hierarchies of motives have been re-oriented – re-oriented by the emotional residue left after the intensely negative experience of being told that they would be playing Around the World. For 36 out of 38 of these prospective teachers, this re-orientation resulted in a motive to not cause their students anxiety took its place as the primary motive at the peak of the hierarchy. This was not a new motive, but rather a motive that promoted up the ranks as a result of their emotional experience. And even after six weeks, and after 50 activities and 400 pages of literature, the concern for student anxiety remained as the primary motive.

These results show, for one of the first times, that emotions are not simply fleeting abstract notions. By anchoring them in the phenomena which spawned them Leont’ev’s Activity Theory is able to show that emotions are robust and powerful contributors to the motives and future action cycle.

AFTERWARD

Although the research presented here, positioned within the participationist tradition, was able to link emotions to motives, it was not able to link emotions to beliefs. The participationist and acquisitionist traditions remain unbridged – for now. But there is still hope. Hannula (2006) has also linked emotions to motives on the acquisitionist side of the divide. Perhaps, in these parallel relationships a bridge can be found.

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BELIEFS AND BROWNIES:
IN SEARCH FOR A NEW IDENTITY FOR ‘BELIEF’ RESEARCH

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Belief research (BR) has contributed to a better understanding of teachers’ acts and meaning making, but is fraught with conceptual and methodological problems. Further, the premise that teachers’ beliefs impact practice is often not confirmed. I compare BR with a conceptual framework, Patterns of Participation (PoP), that shifts the focus from teachers’ beliefs to their participation in a variety of social practices. In order to do so, I first discuss modalities of theory networking and present a general approach to comparing theories and frameworks. The result of the comparison is that PoP shares some of the rationale of BR, but involves a fundamental shift of identity for research on affect, which alleviates some of the problems of BR and is useful for understanding the dynamics of teachers’ contribution to classroom practice.

Lester & Lambdin (1998) draw on two different metaphors to describe developments in mathematics education research. The first, known since the times of ancient Greece, is the story of Theseus whose ship lies in the port of Piraeus. The ship needs repair, and one of the planks is removed and replaced by another. This certainly does not change the ship much; it is still Theseus’ ship. However, over time more planks are replaced, and in the end none of the original planks are left. The question is whether the ship now lying in the harbour is still Theseus’ ship. The other metaphor is about a brownie baker, who gradually changes the ingredients of his delicacies, replacing butter with margarine, whole eggs with an egg substitute, and chocolate with carob. But when does changing the ingredients one by one turn the products on display in the baker’s shop window into commodities that can no longer be called brownies?

Lester & Lambdin use the metaphors to frame a discussion of seven quality criteria for evaluating mathematics education research and – more specifically – for reviewing manuscripts submitted to the Journal for Research in Mathematics Education. In the present context I use them for a somewhat different purpose, namely as a (slightly rhetorical) backdrop for comparing and contrasting two different approaches to research on affect in mathematics education, those of mainstream belief research and of a recent conceptual framework called Patterns of Participation (PoP). The main aim is to discuss the mutual relationships and possible (in-)compatibility of the two approaches, as this may highlight significant developments as well as possible ways ahead for the field of affect in mathematics education. The comparison is made with reference to research on and with teachers of mathematics.

I begin by considering some defining characteristics of any field of study. To stay with the metaphors above, these characteristics constitute ‘the planks of the ship’ or ‘the ingredients of the brownie’, as without them there is no ship or brownie and if changed
dramatically, the result will no longer be recognised as the same as the original. Following from that, I discuss the main characteristics of mainstream belief research and recapitulate parts of a critique of this field. Subsequently, I outline the PoP-framework, before comparing and contrasting the two approaches.

**THEORY NETWORKING**

Research in mathematics education draws on a long list of theoretical imports while seeking to develop a range of what Steiner refers to as homegrown theories (Steiner, 1984). This situation calls for a discussion of the tensions and possible compatibility between different theoretical perspectives (Cobb, 2007; Gellert, 2008; Radford, 2008) and of what Bikner-Ahsbahs & Prediger (2010) call approaches to theory networking. Bikner-Ahsbahs & Prediger order such approaches on a scale of theory integration and position them between the poles of ‘ignoring other theories’ and ‘unifying [theories] globally’. Cobb’s and Yackel’s coordination of constructivist, interactionist, and socio-cultural analyses in their emergent perspective on classroom processes and student learning is a classic example of an approach that may be placed towards the higher end of the scale (Cobb & Yackel, 1996). Networking strategies positioned towards the lower end of the scale include ‘comparing’ and ‘contrasting’ theories or approaches. In this paper I compare and contrast mainstream belief research and PoP.

Bikner-Ahsbahs & Prediger suggest that it may serve three different purposes to compare and contrast theoretical approaches, namely to engage in inter-theoretical communication so as to improve understandings of the theories involved; to function as a less laudable “competition strategy on the market of available theories and theoretical approaches”; and to provide ”a rational base for the choice of theories” (Bikner-Ahsbahs & Prediger, 2010, p. 493, emphasis in the original). My intention when comparing and contrasting mainstream belief research and PoP is to achieve all three, although modifications are needed in relation to the two last purposes. My concern with the second purpose, to promote one at the expense of another, is not based on the premise that one approach is ‘more true’ than the other in any absolute sense. I merely suggest that PoP is helpful in addressing some of the problems that have faced belief research and that it offers a useful and qualitatively different understanding of unfolding classroom events. With regard to the third purpose, the rationality of the endeavour, rationality in the present context merely means making justified decisions for the pragmatic purpose of theorising classroom processes, when those processes are seen from a certain vantage point.

Also considering modes of theory integration, Radford (2008) suggests that theories may be thought of as triples of basic Principles, Methodologies, and paradigmatic Questions. The Principles constitute a hierarchically ordered system of views and statements that “delineate the frontier of what will be the universe of the discourse and the adopted research perspective” (p. 320). The Methodology is not merely the methods used and the reasons for using them, but the process through which particular aspects of or relationships in, for instance, an interview transcript or a video-recorded
lesson become data, that is, worthy of attention in the interpretation. In this sense Methodology is linked to the theoretical stance and the unit of analysis. Finally, the paradigmatic Questions are the ones that initially orient the field and that leave an imprint on the approach taken to other questions that are phrased in subsequent studies that build on the theory. Radford uses the triples, (P,M,Q), to consider ways in which theory networking may be accomplished, and refers, for instance, to when the Principles of one theory are connected with the Methodology of another.

It is beyond the present paper to discuss the controversial question of what is required for a framework to generate or qualify as theory, apart from making the, probably uncontentious, observation that neither belief research nor PoP does so at present. In spite of that, I refer to Radford’s (P,M,Q) triple, among other perspectives on theories and frameworks, in my comparison of belief research and PoP, implying that I also find the triple useful when discussing frameworks that do not qualify as a theory.

COMPARING AND CONTRASTING FRAMEWORKS

In order to address the question of how one may compare or contrast conceptual frameworks and the way they are used in empirical studies, an outline is needed of what a conceptual framework is and how it is related to other aspects of the process of inquiry. According to Miles and Huberman (1994) a conceptual framework “explains, either graphically or in narrative form, the main things to be studied [...] and the presumed relationships between them” (p. 18). While this suggests that conceptual frameworks are somewhat descriptive accounts of the selected key concepts and variables and of their mutual relationships, Eisenhart (1991) focuses more on the reasons for the selection. She argues that a conceptual framework is

“an argument that the concepts chosen for investigation and interpretation and any anticipated relationships among them, will be appropriate and useful, given the research problem under investigation” (p. 209, emphasis added).

For the purposes of this paper, I suggest – more in line with Eisenhart than with Miles & Huberman – that there are three aspects to a conceptual framework. These are F(1): a preliminary understanding of the concepts involved and of the relationships between them; F(2): a (possibly loosely organised) theoretical stance that orients the interpretations of these concepts and other aspects of what Radford calls Principles; and F(3): the overall rationale, that is, a justification that it is worthwhile to engage in the field of inquiry. Such a rationale may be based on practical problems of mathematical teaching and learning, on significant scholarly literature in the field under investigation, or on some combination of the two. The emphasis on F(2) and F(3) in this understanding of a framework implies that different frameworks may not ‘speak the same language’, even when they use the same or similar terminology. ‘Learning’ or ‘development’, for instance, are key concepts that are linked to ‘social interaction’ in a variety of decidedly incompatible frameworks, the differences among them becoming apparent only if the theoretical stance, F(2), and the rationale, F(3), are taken into account. As frameworks develop and are used for empirical purposes, they
may also come to include what Radford calls paradigmatic questions $F(4)$ and aspects of methodology $F(5)$.

In this understanding a conceptual framework orients specific empirical studies by suggesting the key constructs under investigation, as well as the perspective from which they are viewed. This implies that in any particular study the framework is reflexively related to the research questions, to the unit of analysis, as well as to the methodology, that is, to the design and the processes of data generation and analysis (Skott, 2013). This understanding of methodology is much in line with Radford’s interpretation as outlined previously, as it includes M(1): the methods used; M(2): the reasons for using them; and M(3): the question of what data are generated from an empirical situation. The relationships among the framework and other aspects of a field of inquiry are depicted in figure 1.

Figure 1: Frameworks and other aspects of a field of inquiry

It follows that a comparison of research on teachers’ beliefs and PoP should highlight differences and similarities concerned with the three (or five) aspects of the respective frameworks as well as the relationships among the frameworks and the dominant research questions, the units of analysis and the methodology. I begin by describing research on teachers’ beliefs in these terms.

RESEARCH ON TEACHERS’ BELIEFS

Different fields of research specify each of the elements of the proposed comparison to different degrees. Research on teachers’ beliefs, for instance, is basically defined by its key concept, that of beliefs, and studies in the field do not always specify the character of other aspects of the framework used, especially the theoretical stance. The first task, therefore, is through a somewhat abductive approach to suggest the character of this stance based on an outline of other parts of the framework as well as of the unit of analysis, the main research questions, and the dominant methodology. The analysis suggests that research on teachers’ beliefs is framed by an acquisitionist approach.
often inspired by constructivism, and even when informed by more social interpretations of human functioning, frameworks are still inspired by acquisitionism.

**Researching teachers’ beliefs**

Developing from the early 1980s onwards research on teachers’ beliefs aimed both to understand teachers’ actions and meaning making as they relate to classroom practice and to solve or at least alleviate what is generally referred to as ‘the problems of implementation’ (Skott, 2015a). These aims, often somewhat at odds with one another, constitute the basic rationale of the field as it has developed over the last three decades.

The key concept of beliefs is also the main unit of analysis. Sometimes the emphasis is on individual beliefs, for instance about the contents of instruction, while in others it is on structural aspects of belief systems (cf. Zembylas & Chubbuck, 2015). Studies of the latter draw, for example, on Rokeach (1969), a social psychologist whose contributions were made before research on teachers’ beliefs gained prominence. Rokeach developed a framework for conceptualising the relationships between beliefs, attitudes and values. Much later, and working in mathematics education, Goldin (2002) also suggested a conceptualisation of the relationships between different concepts related to the affective domain, namely of beliefs, values, attitudes and emotions. In Goldin’s framework these differ on the three dimensions of how cognitive, how stable, and how “warm” they are. I have suggested elsewhere (Skott, 2015a) that irrespective of the relative emphasis on individual beliefs and on structural aspects of belief systems, the concept of beliefs is still underspecified. This is so even though there is a core to how the concept is used, as beliefs generally refer to those of an individual’s mental constructions that are (1) subjectively true; (2) affectively laden; (3) outcomes of substantial prior experiences; and (4) significant determiners of the individual’s actions and meaning making. Beliefs, then, are mental constructs characterised by considerable degrees of conviction, commitment, stability, and impact (Skott, 2015a).

One set of questions in belief research focuses on the character of teachers’ beliefs about mathematics and the teaching and learning of mathematics as well as about the teachers themselves as learners, teachers, and doers of mathematics. Another set of questions, related to belief stability, is concerned with how beliefs change as teachers engage in teacher education or development programmes or move into full time teaching. It is often assumed that belief change is a long term endeavour that resembles conceptual change in cases when accommodation is difficult to achieve. A third and final set of questions concerns how teachers’ beliefs function in relation classroom practice. Fives & Buehl (2012) argue that beliefs serve different functions, as some filter information, others frame problems, and still others guide action. Also addressing the last set of questions, Thompson (1984) asked:

“1. Are there incongruities between teachers’ characteristic instructional behavior and their professed conceptions of mathematics teaching?
2. How can incongruities between teachers’ professed conceptions and their instructional practices be explained?”
3. Are differences among the teachers in their characteristic instructional practices related to differences in their beliefs and views about mathematics and mathematics teaching?” (Thompson, 1984, p. 107).

Thompson’s paper signalled the introduction of research on teachers’ beliefs in mathematics education, and her questions are paradigmatic in Radford’s sense (cf. the previous discussion) or at least exemplary for the questions asked in the larger part of the field. In this sense they have become part of the framework and orient the overall approach taken in the field of beliefs. The questions are based on the premise that beliefs are the default explanation for classroom practice and explicitly address the corollary that observed incongruities require further explanation (Skott, 2015b).

With regard to methods, it is acknowledged that beliefs are not easily accessed (Kagan, 1990; Richardson, 1996). They are inferred from or attributed to teachers on the basis of different data sources, including questionnaires, interviews, think aloud protocols, and classroom observations. Multiple methods are needed, so as to view teachers’ beliefs from a variety of different perspectives (Kagan, 1990; Schraw & Olafson, 2015). The use of such combinations of methods indicate that the field is generally in line with Rokeach’ remark that a belief is “any simple proposition […] inferred from what a person says or does, capable of being preceded by the phrase ‘I believe that’” (Rokeach, 1969, p. 113). It also implies that the different methods are expected to shed light on the same set of beliefs as viewed from different vantage points.

I suggested previously that $F(2)$, the theoretical stance in belief research, is not always made explicit. However, from time to time reference is made to constructivism and the literature on conceptual change as a sources of inspiration in studies of teachers’ beliefs (e.g. Kagan, 1992). The above outline of the rationale, the unit of analysis, the exemplary questions, and the dominant methodological stance suggests that in general the field is informed by basic constructivist tenets of assimilation, accommodation, and mental disequilibria even when no such reference is made explicit. Belief research, then, extends Glasersfeld’s comment that radical constructivism is based on the premise that knowledge “is in the heads of persons” (von Glasersfeld, 1995, p. 1). In belief research, beliefs that do not qualify as knowledge because of the subjectivity of the truth claim and of its more value-laden character are also conceived of as mental entities located in the head of the individual. It is, then, no coincidence that research on teachers’ beliefs developed in tandem with the constructivist revolution of the 1980s.

In summary, beliefs are considered objectified mental entities that reside within the individual and that take on a life of their own which is independent of the experiences on which they were initially based. In this sense the outline suggests that belief research is based on acquisitionism as a metaphor for human functioning, that is, on a metaphor that “make[s] us think of knowledge as a kind of material, of the human mind as a container, and of the learner as becoming an owner of the material stored in the container” (Sfard, 2008, p. 49).
The problems of belief research

In spite of the progress made in the field of beliefs, it is still a problematic endeavour. The unit of analysis, individuals’ beliefs, is ill-defined and hardly operationalisable; it is acknowledged that the methods do not provide immediate access to these elusive constructs; and one part of the rationale, the expectation of belief impact, is refuted as much as confirmed in empirical studies (Fives & Buehl, 2012). One may wonder why the field is not in a crisis.

As a response to the last of these problems, different scholars have adopted less causal and more dynamic interpretations of the beliefs-practice quandary and reconsidered the expectations of belief stability and belief impact, that is, of the last two of the core characteristics of beliefs (cf. the previous discussion) (Skott, 2015b). Hoyles (1992), Cobb & Yackel (1996), Skott (2001), Sztajn (2003), and Schoenfeld (2011) among others, acknowledge that beliefs about mathematics and its teaching and learning do not necessarily determine practice, and that the degree of influence depends on contingencies arising in classroom and on the relationships among beliefs and other mental constructs. However, these scholars all rely on some form of acquisitionism, albeit in different ways and to different degrees (Skott, 2015b). In this sense they do not challenge the dominant theoretical stance in the field of beliefs.

One part of the background to the Patterns of Participation framework (PoP) described below is also an attempt to address the three problems of belief research that are mentioned above. Doing so, however, PoP adopts a more participatory stance to researching teachers and teaching.

PATTERNS OF PARTICIPATION (POP)

As in research on teachers’ beliefs, a dominant part of the rationale of PoP is to develop understandings of teachers’ acts and meaning making as they relate to classroom interaction. Also, and in line with recent studies of beliefs, PoP addresses the problem of the lack of support to the congruity thesis. However, in contrast to mainstream belief research PoP finds the conceptual and methodological problems of the field of beliefs equally disturbing and also suggests that the acquisitionist stance is ill-suited to be combined with recent approaches to the study of teachers’ professional identities (Skott, 2013). Further, a number of recent and somewhat grounded studies of belief-practice relationships have challenged the acquisitionist underpinnings of belief research (Skott, 2004, 2009; Wedege & Skott, 2006).

PoP is, then, an attempt to develop a more participatory account of the functioning of teachers in mathematics classrooms. The key concepts of the framework include those of practice, participation, and figured worlds from social practice theory (Holland, Skinner, Lachicotte Jr, & Cain, 1998; Lave, 1988, 1997; Wenger, 1998) and of objectification and discourse in Sfard’s recent theory of commognition (Sfard, 2008)\(^1\).

\(^1\) For a more comprehensive introduction to the background of PoP, see Skott (2013).
However, both social practice theory and the theory of commognition need to be supplemented to be useful for PoP-purposes. While the understanding of practice and participation in social practice theory is not a “displacement of the person” (Wenger, 2010, p. 181), I suggest that there is a need to re-centre the individual both in social practice theory and in Sfard’s account of learning. Further, Sfard’s emphasis is on well-structured, socio-cultural discourses (e.g. mathematics) and needs to be supplemented with a concern for the role of other practices and figured worlds in and for immediate social interaction when considering less structured practices such as teaching. I have found the I-me dynamic in the symbolic interactionist notion of self helpful for both purposes (Blumer, 1966, 1969; Mead, 1913, 1934).

According to symbolic interactionism interaction is based on the human ability to take the attitude of individual and generalised others to ‘objects’ in their environment. As we act, we view the objects of our interactions as well as ourselves through the eyes of other interlocutors, both present and absent, and we modify our actions based upon their indications about what is important and why. Consider, for example, a teacher working with a group of students, who are trying to substantiate a mathematical conjecture, but find it difficult to do so (cf. Skott, 2015b). The teacher anticipates and interprets the words, the tone of voice, the raised eyebrows, etc. of the students in the class. However her contributions to the interaction may change if she, while engaged in a mathematical discourse to assist the students, also orients herself towards a proposal for educational reform promoted by her teacher education programme; positions herself in a team of teachers whose collaboration focuses on the well-being of individual students rather than on their subject-matter learning; and attempts to document her mathematical expertise, as her subject matter competence was recently questioned at a PTA meeting. In symbolic interactionist terms, the teacher takes the attitude of different individual and generalised others (the students, the teacher education programme, her team, the parents) and draws upon and renegotiates the meaning of the related social practices and discourses in the process. The theoretical stance of PoP, then, coordinates key understandings from social practice theory and commognition with others inspired by symbolic interactionism.

These theoretical sources of inspiration and the intention of re-centring the individual imply that the key unit of analysis is the individual in multiple practices and figured worlds. Exemplary research questions for PoP include: What roles, if any, do a teacher’s tales of herself as a professional play for how she engages the students in endorsing mathematical narratives? How, if at all, does her relationship to educational discourses (e.g. the reform) modulate how she and her students negotiate the meaning of mathematical proficiency? How, if at all, does collaboration with her colleagues play a part in how she positions herself in relation to the students? (Skott, 2013).

The methods used in PoP are partly the same as the ones used in belief research and include interviews with and classroom observations of the teacher in question. As deemed necessary in the particular study, these are supplemented with for instance interviews with the teacher’s colleagues and the school leadership and with
observations of informal exchanges in the staff-room, of meetings in teams of collaborating teachers, and of teacher development sessions (Skott, 2011, 2013). However, in belief research different methods are used for triangulation purposes, that is, to provide “multiple lenses on the same phenomena” (Schoenfeld, 2007, p. 87). In contrast, PoP does not use multiple methods because they are expected to shed light on the same construct of beliefs, but exactly because they provide at least some access to how the teacher engages in different practices and figured worlds. An interview conducted in the researcher’s office, for instance, may engage the research participant in an educational discourse on a reform agenda, while a team meeting may constitute a very different practice that is only remotely, if at all, related to current reform initiatives. Classroom observations are the key mode of data generation and the teacher’s acts and meaning making are interpreted in terms of the role of, for instance, the reform discourse and of her collaboration with her colleagues for her contributions to the practices that emerge at the instant. In symbolic interactionist terms, these and other discourses and figured worlds may (and may not) play the role as significant generalised others for the teacher in question as classroom interactions unfold. It is apparent, then, that while the methods used, $M(1)$, in belief research and in PoP are largely the same, the reasons for using them, $M(2)$, are very different and so are the data that are generated from the empirical situation, $M(3)$.

**SUMMARY AND CONCLUSIONS**

It is apparent from the outlines above that there is a common element to the rationale of mainstream belief research and PoP: Both attempt to understand individual teachers’ acts and meaning making. In spite of that, the theoretical stance differs significantly between the two fields. Further, key notions (e.g. practice) are interpreted differently, and while belief research sees meaning-making as based on objectified mental constructs, PoP adopts a processual perspective and considers it a matter of reengaging in other social practices in view of the interactions that unfold at the instant. These differences lead to qualitatively different units of analysis (objectified mental constructs vs. person in multiple practices) and paradigmatic research questions, and although the methods are similar, they are used for decidedly different purposes.

As mentioned before, a number of attempts have been made in belief research to address the beliefs-practice quandary by adopting less causal and more dynamic interpretations of the relationship between teachers’ beliefs and instructional activity (Lerman, 2001; Schoenfeld, 2011; Sztajn, 2003). Generally, these attempts do not challenge the basic rationale or other parts of the acquisitionist framework. Returning to the metaphors in the introduction of this paper, they may replace a plank or two in the ‘ship’ of belief research or change a single ingredient in the recipe of the ‘brownie’. Belief research, however, is still belief research. My argument is that PoP is not. The framework is fundamentally changed and so are the exemplary research questions and the unit of analysis, and the methodology is altered to the extent that what constitutes data is different even if the methods used may be superficially similar.
I argued previously that there are two sides to the rationale of belief research, as it aims both to understand teachers’ actions and meaning making and to solve ‘the problems of implementation’. PoP research shares the first aim, and my colleagues and I have used the PoP framework for empirical purposes and argued that it offers a dynamic and contextual perspective on the acts of teaching that cannot be developed in mainstream belief research (Palmér, 2013; Skott, 2013; Skott, Larsen, & Østergaard, 2011). One may wonder, how PoP relates to the other part of the rationale of belief research, that is, to the ambition of solving the ‘problems of implementation’. From a PoP perspective ‘implementation’ is a problematic metaphor, as it carries connotations of the smooth execution of a plan or idea. But if practices in schools and classrooms are emerging and continuously renegotiated, a reform initiative, for instance as discussed in a teacher development programme, can hardly be expected to be ‘implemented’. It may function as yet another generalised other that teachers may draw on as classroom processes unfold. But there is little reason to expect a priori that the initiative dominates the acts of teaching, let alone the communal practices that evolve in the classroom.

This may seem a disappointing conclusion. After all, mathematics education research is expected to contribute both to the development of new understandings of teaching-learning processes and to the further development of these processes. The comments above suggest that PoP has little to offer in terms of the latter of these intentions. However, PoP does not imply that mathematics education research should not attempt to influence practice. It does suggest, though, that modest expectations should be set with regard to impact. But this merely recognises that classroom practices are emerging and social, not the outcome of ‘belief enactment’. In this sense, PoP offers an explanation why belief research has not fulfilled the promise of its founders and solved ‘the problems of implementation’.

REFERENCES


“... AND THEY LIVED HAPPILY EVER AFTER”

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INTRODUCTION
As a celebration of the 20th MAVI conference, we had three speakers: Günter Törner, Peter Liljedahl and Jeppe Skott. They were given the following task: what are the characteristics of research in affect and mathematics education? Where are we now? Where are we going? These questions resulted in three very different presentations. I had the pleasure to summarize and synthesize the talks and the discussions that followed.

TÖRNER’S TALK
Törner gave a historical exposé of research in beliefs, and he started his presentation by stating that beliefs are no longer a hidden variable. He continued by saying that we are surrounded by beliefs and that research has concluded that we are highly influenced by beliefs. However, as a concept it is not easily defined and it is still a fuzzy construct. In his paper, Törner lists various definitions different researchers have provided in order to capture the phenomena. The sheer number of definitions could show a lack of respect of history in beliefs research if every researcher feels he or she needs to formulate his or her own definition. (The author of this paper is guilty of this as well.) On the other hand, the sheer number of different definitions indicate that this is a complex area. If researchers constantly find new aspects of a concept, surely it must be part of the development of the theoretical stance. Perhaps we have not reached the final definition of beliefs (and other related concepts)?

And, the vast quantity of different definitions influence methods of data collection and methods of analysis as well. How does one study another person’s “world view” in contrast to ‘Spaghettibundels’ (belief bundles)? Is it possible to make a distinction at all? In the paper, Törner ask about the ‘solid findings’, but can we really talk about solidity when it comes to other peoples ‘world view’? Törner raises a few methodological aspects when listing homework to be done. He writes:

Seldom authors of belief papers make their anchoring in a specific theory transparent.

When we deal with a fuzzy construct, which has no single definition, that stems from numerous affective parts such as motivation, emotion, values etc. and is also highly connected to personal experiences and thereby cultural contexts, we, as researchers, need to be transparent in terms of theories and both methods of data collection and method of analysis. This is especially important when studying potentially the most
complex research topic of all: beliefs and behaviour. Törner’s conclusion was that this is still an open question with no easy explanation. Törner also would like to see beliefs as modules of larger theories. Many theories involve beliefs, but beliefs are not often discussed explicitly. Perhaps there is a reason why?

**LILJEDAHLL’S TALK**

In Liljedahl’s talk, we see an attempt to solve some of the problems that stems from the acquisitionist’s construction of the concept of belief. He would like to offer a bridge. The building of the bridge uses emotions as a starting point to move towards a participationist view using Activity Theory which was formulated by Leont’ev. Through an example, he showed how Tara changed the priorities of motives as a teacher, after participating in a lesson that had the goal of highlighting the affective domain and mathematics education. The results were very convincing: solid findings anchored in well-defined theories. This was a study of actions, just as Törner had asked for. Liljedahl concludes:

> These results show, for one of the first times, that emotions are not simply fleeting abstract notions. By anchoring them in the phenomena which spawned them Leont’ev’s Activity Theory is able to show that emotions are robust and powerful contributors to the motives and future action cycle.

However, in his presentation, the focus was on emotions and (through needs) motivation, but that was no space for beliefs. Liljedahl did state that the absence of evidence is not equal to the evidence of absence but before he could take a second breath, Tirosh as a response asked two questions: (1) What is underneath the concepts of emotion, motivation etc?; and, (2) Do you need beliefs to explain other matters? These questions reintroduced the complexity of the matter. We cannot simply throw away the acquisitionist perspective and beliefs remain a useful construct.

Another issue soon was arisen: Where was the mathematics in Liljedahl’s presentation? The relatedness, as a scientific criterion, must then be low (c.f. Kilpatrick, 2003). Maybe it was due to the question that was initially posed? By focusing on emotions and using theories based on this, the cognitive aspects of beliefs – the subjective knowledge – becomes more or less invisible. What is certain is that Liljedahl’s presentation showed how intricate beliefs are and also emphasized the need to explore other affective concepts in order to gain a better understanding of beliefs. As Liljedahl writes in his paper:

> The participationist and acquisitionist traditions remain unbridged – for now.

The homework is still valid.
SKOTT’S TALK
The starting point for Skott’s talk is the theory of Patterns of Participation. The main question raised is if and how different theories of framework may be connected. Very thoroughly, Skott answers this question by comparing and contrasting mainstream belief research and Patterns of Participation. He shows that the latter is helpful when addressing some of the problems that have faced belief research (c.f. Törner’s list of homework). Patterns of Participation uses a processual perspective, meaning it is dynamic and includes contextual factors. Compared to most belief research, Skott shows the relationship between the theoretical stance of Patterns of Participation and research questions, methodology and unit of analysis. Through this, Skott shows that Patterns of Participation is no longer belief research. He writes:

The framework is fundamentally changed and so are the exemplary research questions and the unit of analysis, and the methodology is altered to the extent that what constitutes data is different even if the methods used may be superficially similar.

And he is right, at least to some degree. Patterns of Participation offers different tools to study human interaction. It is indeed a valuable contribution to belief research. But at the same time it is a perspective that limits the researcher. Skott raises this issue himself:

From a PoP perspective ‘implementation’ is a problematic metaphor, as it carries connotations of the smooth execution of a plan or idea.

It is indeed a dilemma. What it is the purpose of mathematics education research if we remove implementation? What is the point if we cannot predict results? Another issue is the one-sided focus on language, and I will here turn to Barad (2003) and posthumanist performativity. 12 years ago, she wrote:

Language has been granted too much power. The linguistic turn, the semiotic turn, the interpretative turn, the cultural turn: it seems that at every turn lately every “thing”—even materiality—is turned into a matter of language or some other form of cultural representation (Barad, 2003, p.801).

Barad’s (2003) conclusion is that matters matter. Taking this into account, we can see that ‘matter’, just as cognition was lacking in Liljedahl’s paper, is missing from Patterns of Participation. Shall we really discard cognition as a human factor?

SUMMARY
As a member of the audience, it was a pure pleasure to listen to these three talks. They all had their strengths: Törner’s paper based on a huge body of research, Liljedahl’s paper showed great inventiveness, and Skott provides a theoretical framework where the acquisitionist perspective has failed. But they also have their weaknesses. Törner’s paper shows that is an overabundance of definitions regarding the concept of belief, and there could be a simple answer why: if beliefs are a messy construct, perhaps there
is something wrong with the concept? In Liljedahl’s presentation there was no space for beliefs, but could this be due to not incorporating cognitive matters from the beginning? Skott’s paper provides a solid theory, but if you can’t implement your results and generalise, what is the point of doing research? My conclusion is that these three papers show us why we need different theories: no single theory can answer every question. If we are going to understand interesting, complicated problems, we cannot profess ‘one truth’. This means that we, just as Törner wrote, need to be transparent about theories and methods. As Österholm (2011) has already concluded:

it is of interest to examine if and how a change of perspective would affect empirical studies about beliefs, as a way to develop belief research in general, regarding both theoretical aspects and empirical results (p.74).

Until we have found the bridge that Liljedahl described, we need both perspectives since they complete each other. And, instead of taking sides as if they are partners in a bad marriage, we need constructive discussions about scope and limitations. And in order to create such discussions, we need the MAVI community to continue the work in the affective domain.

References


CHARACTERISING PARENTS’ UTILITY-ORIENTED BELIEFS ABOUT MATHEMATICS

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Considering the overall goals and purposes of secondary school mathematics, the question arises to which extent parents’ mathematical understanding matches these goals and purposes. As one attempt to resolve this question, a small scale study on parental utility-oriented beliefs of mathematics is conducted in which parents of fifth graders’ from a German Gymnasium took part. A typology is deduced from the gathered data in order to generate a systematic approach to analyse and describe the parents’ utility-oriented beliefs about mathematics. After identifying interrelationships between parental emotional dispositions towards mathematics, the paper ends with a discussion of results and conclusions for future initiatives which aim at encouraging and empowering parents to support their children’s mathematical education.

INTRODUCTION AND THEORETICAL FRAMEWORK

The critical role parents play in their children’s mathematical development in secondary school is undeniable (Desforges & Abouchaar, 2003; Hoover-Dempsey & Sandler, 1995). In general, parental support in mathematics happens through assistance with school related issues, for example, with homework and test preparation or in everyday conversations concerning mathematics related topics. In such situations parents’ objective (formal, public) knowledge, as well as their subjective (experience-based) implicit knowledge and emotions concerning mathematics, which are here understood as their mathematics related beliefs (Pehkonen & Törner, 1996), influence parental behaviour and the way they approach a mathematical situation or support their children in their educational process. These mathematical beliefs describe the context in which parents perceive and deal with mathematics (Grigutsch, Raatz & Törner, 1998).

A frequently used description of mathematical beliefs is the characterisation by Törner & Grigutsch (1994) into three components: the scheme and formalism aspect, as well as process aspect. Later, Grigutsch, Raatz and Törner (1998) add the usefulness of mathematics as another important component to their concept. This specific aspect is more precisely differentiated by Maaß (2006) whose concept of utility-oriented beliefs builds a theoretical basis for this paper. She distinguishes three dimensions, a pragmatic, a methodological, as well as a cultural relevance.

- Within the dimension of pragmatic relevance, mathematics is seen as (personally) useful to understand and manage environmental as well as professional situations. The following three steps are considered to exemplify a pragmatic relevance:
Relevant are those situations in which mathematics are immediately needed in the current life or the foreseeable future.

Relevant are those situations in which mathematics assists one in gaining an understanding or in scrutinizing one’s direct surrounding.

Relevant are those situations in which mathematics assists one in gaining an understanding or in scrutinizing essential aspects of the world.

Within the area of methodological relevance, mathematical competencies and knowledge are understood as basic life skills, such as modelling and problem-solving skills or competencies to critically reflect about things, to communicate with other people supported by mathematics and to cope with new situations.

Within the sphere of cultural relevance, mathematics plays a crucial role for the development of society. A worldview, from a modelling perspective, is adopted which includes critical evaluation of models and the experience of fundamental boundaries for mathematizability.

This categorisation constitutes a theoretical background for a more detailed analysis of beliefs related to the usefulness of mathematics. However, it does not claim to be comprehensive.

At the same time the categorisation of utility-oriented beliefs resembles an understanding of mathematics that can be found in the description of the overall purposes and goals of secondary mathematics education in the German scholastic standards of mathematics for the secondary school certificate (Kultusministerkonferenz 2004, p. 6). Therein three so called fundamental experiences are defined to which (more or less) direct commonalities to the categorisation of utility-oriented beliefs can be found. Those links are outlined in brackets after the citation of each fundamental experience.

- Perceive and understand technical, natural, social and cultural phenomena and processes with the help of mathematics and assess them based on mathematical principles (resembled in the cultural and pragmatic relevance).
- Perceive and understand mathematics (...) in its relevance for the description and solution of intra- and extra-mathematical tasks and problems (focussing on extra-mathematical tasks there are parallels to the pragmatic relevance of mathematics).
- Gain general problem-solving skills through the solution of questions and problems with mathematical methods (commonalities with the methodological relevance).

Hence, within in the scope of exploring a parental understanding of the overall purposes and goals of secondary school mathematics, one promising approach lies in the investigation of utility-oriented beliefs, as they are elaborated by Maaß (2006). By no means do I want to neglect the role of other aspects of mathematics related parental beliefs, but attempt to reasonably limit the focus of this paper. In this context, the
research questions can be specified: Which utility-oriented beliefs do parents have? Which spectrum do these utility-oriented beliefs describe? Are there patterns or interrelationships with other personal variables?

**METHODOLOGY**

The research on parental beliefs takes place within the context of a family math project in which fifth graders of a German Gymnasium and their parents take part on a voluntary basis. The project is accompanied by a pre-post study design which aims at investigating parental beliefs about mathematics, its teaching and learning as a whole. The data presented here was acquired in the pre-study through a questionnaire that consists of open-ended questions. The parents were instructed to answer the questionnaire before their participation in the family math project and more precisely they were asked to take enough time and to complete the questionnaire without disturbances. In order to minimise distortions through social desirability, the participants remained anonymous.

Following an interview manual, the open questionnaire accepts free answers and at the same time gathers as diverse as possible information about utility-oriented parental beliefs. Since the original research by Maaß (2006) was developed for students’ beliefs, this study is only build on those items which are relevant for parental beliefs as well. These are:

- **What is mathematics for you? What are typical features of mathematics?**
  - This item relates to a more general understanding of mathematics. It serves especially to substantiate the interpretations of the parents’ utility-oriented statements.
- **How can you benefit from mathematics in your life?**
- **In your opinion, what role does mathematics play in the development of our society?**
- **Exemplify some situations, in which mathematics is applied to solve a problem.**
- These items can directly be related to the three utility-oriented beliefs of mathematics, the pragmatic relevance, the cultural relevance and the methodological relevance.

Apart from utility-oriented beliefs about mathematics, personal information such as age, gender, education level, vocational training and occupation as well as parents’ motivation for participation in the family math project and their emotional disposition towards mathematics are ascertained by the questionnaire. Those additional items provide further information for the analysis of the interrelationships between parental attributes and their utility-oriented beliefs.

**Analytical approach**

The analytical framework for the open data sources is characterized by principles of qualitative content analysis (Mayring, 2005; Kuckartz, 2014). The analytical aim is to
obtain a structured picture of the parents’ utility-oriented beliefs by developing code rules and a coding system. First the data is reduced by a deductive application of categories which have been elaborated by Maaß (2006), namely the pragmatic relevance, methodological relevance and cultural relevance. Subsequently, this first categorisation is specified by an inductive development of sub-categories with the focus on capturing the diversity and complexity of examples given in parental statements. Since the interpretability of statements can be subjective, the foundation of coding by an earlier defined and discussed system of categories benefits objectivity. This is especially essential because, due to personnel constraints the researchers were unable to collaborate on the initial coding of the data. Only afterwards some non-specific statements could be discussed and clarified among the team.

Following Maaß (2006) and in order to reduce the complex reality of parental statements about utility-oriented beliefs, a typcast procedure is applied to the data. After coding the data as described above, cases were compared, contrasted and a typology deduced. The main focus of this procedure is that the emerging types are internally as homogeneous as possible and externally as heterogeneous as possible (Kuckartz, 2012). Finally, a basic frequency analysis was employed on the data, as well as an examination of the interrelationships was performed. Due to the fact that the present sample is not representative, the quantitative information must not be overestimated.

Using a questionnaire methodology with open items, the overall problem occurs that only conscious beliefs can be reached. When a parent does not express a current state of a belief, it does not mean that this parent does not subscribe to such a belief. It simply reveals that those beliefs are not purposefully expressed, rather, they are unconscious and less marked.

RESULTS

Altogether 35 parents participated in the first workshop and completed the questionnaire beforehand. Due to the fact that some of the participants did not fully complete the questionnaires, only the data from 33 parents can be considered for the analysis, among these 20 mothers and 13 fathers. The parents ranged in age from 36 to 52 years old. According to the catchment area of the school as well as educational and occupational variables, the families in the project belong to a rather high socio-economic segment of the population.

A clear distinctiveness in the parents’ statements regarding the items about utility-aspects suggests the development of a typology. However, during coding, it became clear that the category of methodological relevance could not be considered as a distinctive feature. Parental responses most likely contained only implicit references to this dimension. Hence, it is hard to distinguish methodological relevance without over-interpreting the parental comments. For future research, it should be considered to reformulate the open questionnaire or to add an additional item which addresses the methodological relevance in a more direct way.
Thus, as with Maaß (2006), the deductively given categories of pragmatic relevance and cultural relevance are selected as features for the typology. Within these two dimensions four distinct levels are deduced from the data. Those four levels are named by no or explicitly restricted relevance, basic relevance, distinct relevance and special relevance. This distinction is subjective in its nature since it is based on interpretations and assessments done from personal perspectives on mathematics which may not necessarily coincide with other perspectives.

For reasons of transparency, the four levels of relevance are defined for each utility-dimension, referring to characterisations given by Maaß (2006). A graphic illustration follows including a quantitative analysis of the coded data. Subsequently, some examples are given in order to substantiate the typology and level-definitions.

The pragmatic relevance can be divided into the following four levels:

- **No or explicitly restricted pragmatic relevance** – Parents do mention examples from general areas of practice in everyday life which can be related to the possession of basic numeracy, but beyond that restrict the relevance of mathematics explicitly to those contexts.

- **Basic pragmatic relevance** – The pragmatic relevance is shown through examples from general areas of practice in everyday life, for example, monetary transactions or measurements of weight which concern solely basic mathematical skills, mainly basic numeracy.

- **Distinct pragmatic relevance** – The usefulness is reflected in diverse examples referring to more complex mathematical contexts from and mechanisms beyond everyday life or in the description of the omnipresence of mathematics in life.

- **Special pragmatic relevance** – A vital relevance of mathematics in life is not only mentioned through diverse examples, but in a more universal manner referring to mechanisms and mathematical models. Additionally, the competence to understand or scrutinise essential aspects of the world through mathematics is described. In this sense, mathematics enables a critical world view. In this type a diverse methodological relevance becomes apparent as well.

Similar to the pragmatic relevance, the cultural relevance can also be divided into four levels as well:

- **No or explicitly restricted cultural relevance** – The cultural relevance of mathematics is explicitly restricted or not even mentioned at all.

- **Basic cultural relevance** - A cultural relevance manifests at the utmost in a relevance for scientific subjects itself, e.g. STEM fields or economics, but without drawing the connection to social developments.
Distinct cultural relevance – A distinct cultural relevance is expressed through a diverse spectrum of examples beyond STEM fields and economics or in the description of mathematics as a necessary foundation for cultural developments. A modelling perspective does not emerge entirely.

Special cultural relevance – A so called modelling perspective is adopted in which someone recognises not only the vital connection between mathematics and social development, but explicitly reveals an understanding of use and misuse of mathematics in society, as well as the limits of mathematizability.

Consequently, the feature space of the typology consists of a 4x4-matrix with 16 homogeneous types, though not every type of those 16 can actually be retrieved in the collected data. The seven different types which appear are displayed along with their absolute frequency in Figure 1.

Figure 1: Typology of utility-oriented parental beliefs with the absolute frequency

In the following, exemplary parental statements are displayed for each type, in order to substantiate the typology. Therein the author of the answer is indicated by M for mother and F for father combined with a respective number.

Type I – Parents explicitly neglect or restrict a cultural and pragmatic relevance, as reflected in the following quotation:

I don’t use [mathematics] at all, except for simple calculations in everyday life.

For society [mathematics has] no relevance, for technical developments a high relevance (F24).
Type II – Parents, who are ascribed a distinct pragmatic relevance, mention examples which refer to more complex mathematical skills and mechanisms, prevalently from professional contexts. However, they give no statement related to the item on cultural relevance as the quotation below displays.

[Mathematics is] a daily tool for my job [as a controller]. [It provides] possibilities to visualise complex problems.

[Subject areas, in which mathematics helps to solve problems, are] statistics and price calculations (F17).

Type III – In this example a basic pragmatic relevance is verbalised through examples which refer to basic numeracy, as well as a basic cultural relevance:

[Mathematics] is helpful in several fields, like finances etc.

[Mathematics has an] important meaning for economics and future research (M17).

Type IV – A distinct pragmatic relevance is expressed through diverse examples based on more complex mathematics (from a professional context), as well as a distinct cultural relevance in mathematics. Here, it is described as an indispensable basis for scientific and economic contexts and with this for development in society.

[I benefit from mathematics] both professionally and privately. Professionally [mathematics is used] in the development and judgement of operating numbers for business management and privately in cash-based accounting etc..

[In context of society] mathematics is the basis for all scientific and economic contexts and therefore indispensable for future social developments.

[Some situations, in which mathematics is applied to solve a problem are] the calculation of the physics of driving, weather forecasts, multimedia, business studies etc. (F19).

Type V – In this type, the view of parents does not exceed a basic pragmatic relevance based on handling numbers and basic mathematical skills for everyday life. At the same time a special cultural relevance emerges in the form of a modelling perspective.

[Wherefore mathematics is used in life?] to master processes of everyday life.

Mathematics hides behind every product and economic processes, which form the basis for the development of society. The ignorance of the population and politics about the nature of national debts will bring social development to a standstill.

[Situations, in which mathematics is used to solve problems, are] rule of three in everyday life [from cooking to shopping], the statistical analysis of students achievements, the calculation of area and volume. (V4).

Type VI – One father can be ascribed to this type since he mentions a special cultural relevance reflecting a modelling perspective in his remarks, but only a distinct pragmatic relevance. He points out diverse examples based on complex mathematical skills. Though, he does not refer to the understanding or questioning of central aspects of the world through mathematics.
To be precise, [I benefit from mathematics] in the management of my company. It is about financial controlling and related parameters (...). Without statistics a globalised society would be inconceivable. I think it is of special relevance to visualise such differences and social changes. [It helps] to draw conclusions.

[Situations, in which mathematics is used to solve problems, are] the financial crisis and the subsequent recovery, research and development, exploration of demoscopic topics, in the field of general optimisations (...) (F26).

Type VII – Here a special pragmatic relevance is displayed, especially with references to the understanding or questioning of essential aspects of the world through mathematics. Furthermore a special cultural relevance becomes apparent in the form of a modelling perspective.

In daily life [mathematics] provides readability and countability, without [mathematics] you can’t compare or plan. Furthermore, it expands reference frameworks and awareness. It facilitates journeys into intellectual worlds. It enables insight and validation through the understanding of formalisms and causal relations (interest-calculation, theory of probability, calculation of radioactivity, logic) and thereby it ultimately makes democracy, in a political sense, possible (F12).

Searching for interrelationships to other personal characteristics, one significant relationship occurred between the type of utility-oriented beliefs and parents’ emotional dispositions towards mathematics. One item, right at the beginning of the open questionnaire, aimed at this aspect of the parents’ mathematics related beliefs which reads as follows: Do you like mathematics? If so, why? If not, why not?

Parents’ responses to this item were categorised into three types which are positive emotional disposition (Y: Yes, I do like mathematics), negative emotional disposition (N: No, I do not like mathematics) and variable emotional disposition (D: It depends). In Figure 2 the typology of utility-oriented parental beliefs is supplemented by the absolute frequency of emotional dispositions which were assigned to parents of the respective type.
DISCUSSION AND CONCLUSION

To understand the whole picture, some facts need to be mentioned before the discussion of results. First, the sample consists of parents with a high socio-economic status and second, they participated in the family math project and in the questionnaire voluntarily. About 110 parents had the opportunity to participate, but 35 actually did. All of these parents have a fundamental motive for participation in the family math project. Consequently, the sample is selective by its nature. However, taking into account all of the conditions, the data still allows some interesting insights.

As it becomes apparent in Figure 1, most attributions of utility-oriented beliefs (25 out of 33) are situated on the diagonal of the typecast which moreover is fully populated. In other words, the manifestation of both dimensions, the pragmatic and cultural relevance, prevalently occurs on the same level. This suggests two assumptions. The first assumption would be that splitting utility-oriented beliefs into a cultural and a pragmatic relevance is redundant since both dimensions account for the same thing. However a contrary indication is that there actually are 7 parents (Type II and V) whose statements display a clear difference in levels of pragmatic and cultural relevance. Those deserve attention and with that, a distinction of pragmatic and cultural relevance is indispensable. This leads to a second assumption. Levels of parents’ utility-oriented beliefs seem causally linked to each other. That means, gaining an understanding in one dimension of utility-oriented beliefs commonly, but not in every case, leads to a growth in the understanding of the other dimension.

All in all, there are 12 parents who express distinct or higher rated beliefs in both dimensions, pragmatic as well as cultural relevance (Type IV, VI and VII).
beliefs can be related to an understanding of mathematics that basically fits the *fundamental experiences* of mathematics education. 10 of those 12 parents clearly express an affinity for mathematics. Two parents mention that it depends on the topic whether they like mathematics or not. On the opposite end there are 14 parents who reveal a basic view or less in both dimensions (Type I and III) and 7 parents in one of those two dimensions (Type V and II). These levels do not fit the *fundamental experiences* of mathematics education. It does not mean that those parents would not agree to a *distinct* or *special relevance*, but that those beliefs are not mentioned, rather unconscious or less marked. Additionally, nearly half of those parents (10 out of 21) express a clear aversion towards mathematics and furthermore 7 parents mention that it depends on the complexity of the problem as to whether they like mathematics or not. As reasons for their aversion towards mathematics, parents mention the rigidity and abstractedness of mathematics, but mostly bad experiences in school and difficulties in understanding mathematics. Those affect-related frequencies stand in high contrast to the above discussed parents of Type IV, VI and VII. As the results show, parents’ emotional dispositions towards mathematics and the level of utility-oriented beliefs which they express frequently go hand in hand.

Considering these results for the design of an initiative that pursues an active integration of parents in their children’s mathematical education, it becomes clear that especially parents who state an aversion towards mathematics need special attention since those frequently state utility-oriented beliefs on a basic level or less. With regard to reasons parents mentioned for their aversions, some starting points for such an initiative can be derived. Different possibilities for positive mathematical experiences should be enabled, that means a surrounding where parents can gain new insights into mathematics, apart from rigidity and abstractedness, as well as a feeling of mathematical competence. Additionally, several relevant aspects should be taken into account in the thematic design. With a closer look to the above presented results, those initiatives should incorporate a *special pragmatic relevance*, which refers to the understanding and questioning of essential aspects of the world through mathematics, and a *special cultural relevance* in terms of emphasising a modelling perspective. Since explicit statements related to a *methodological relevance* of mathematics could barely be found in the data, this relevant dimension needs be considered as a key aspect in initiatives for parental integration as well. A methodological relevance of mathematics includes, for example, modelling and problem-solving skills, as well as competencies, supported by mathematics, that will assist parents in communicating with other people. Fortunately, the here distinguished three relevance aspects are not mutually exclusive. In fact they are complementary and could all be addressed at the same time.

In this context, the question that remains is whether an initiative that deliberately incorporates these factors could achieve the development of utility-oriented parental beliefs. If so, to which extent?
REFERENCES


This paper questions the validity of a methodological framework aimed at codifying the turbulent undercurrents that characterize students’ group interactions during mathematical activity. Emotions, motivations, will, pleasure, self-confidence, drive the mathematical activity and frame cooperation with others. We asked several teachers to interpret all these affective moves looking at two excerpts codified according to our methodological framework. We discuss consistencies and inconsistencies between teachers’ personal interpretations. The result is a deeper understanding of the group interactions that we had analyzed.

INTRODUCTION

Roth and Radford (2011) argue that learning occurs in and through relations with others driven by collectively motivated activity, and emotions are necessary for the activity and responsible of its development. We (Liljedahl & Andrà, in press) propose a model to codify the students’ interactions, looking at their utterances (see also Sfard and Kieran, 2001), their gazes (with different degrees of intensity), and inferring the emotional dispositions with respect to the activity they are engaged in. Students’ gestures, postures and glances are constitutive components of the meaning making process. Students do not only express and communicate their ideas through the movements of their hands, they do not only stare at each other in order to catch the others’ attention, but the ideas that emerge from the activity are in their gestures and glances—to the point that if we discard these elements we miss many relevant facts (Andrà & Liljedahl, 2014). We also used fictional writing as a way of synthesizing our interpretations of students’ thoughts and will. Following Hannula's (2003) techniques and intentions, we came to confirm that in order to open a window on the students’ inner world it is necessary to repeatedly, patiently, and carefully look at their interactions, their words, and their postures. Gazes give us insights into this inner world and allow us to write a version of the inner monologues of each participant. How authentic are such inner dialogues? Can other, different fictional inner dialogues be written? To what extent, and on which issues, can all the possible fictional inner dialogues agree? To which extent can they disagree? In order to answer to these questions, we have asked several teachers to write their own fictional inner dialogues with respect to the same excerpt, and in this paper we report on
findings from these writings. Our hypothesis is that looking at different interpretations of the same phenomenon from diverse perspectives enhances a deeper understanding of the phenomenon itself.

THEORETICAL BACKGROUND

Two levels characterize our theoretical background: the level of the observation/interpretation of students’ interactions, and the level of the teachers who interpret these interactions.

Students’ interactions

Interactive flowcharts were introduced by Sfard and Kieran (2001) as a way to capture “two types of speaker’s meta-discursive intentions: the wish to react to a previous contribution of a partner or the wish to evoke a response in another interlocutor” (p.58). A conversation can be coded as being comprised of a series of arrows aimed at specific people. The scheme follows for two basic structures: (a) a vertically or diagonally upward arrow is called a reactive arrow and points towards a previous utterance; (b) a vertically or diagonally downward arrow is called a proactive arrow and it points towards the person from whom a reaction is expected. Sfard and Kieran (2001) developed this scheme to code conversations between two people and Ryve (2006) extended it to account for more than two people. Table 1 in our example reads as follows: M in 1 makes a proactive statement to L and D, D reacts in 2, and so on. In our earlier research (Liljedahl & Andrà, in press), we found it was necessary to consider not only the flow of conversation, but also who the participants are looking at. As such, we introduce a new set of arrows, meant to represent where someone is gazing during each utterance. We use red arrows to represent the speaker and blue arrows to represent non-speakers. In Table 1, for example, M looks at the paper in 1, D looks at the paper in 2. On the basis of the interactive flowchart, we asked several teachers to write a fictional inner dialogue from each participant to the activity.

Fictional writing is a methodology articulating our interpretations of the inner monologue of the student, creating likely impressions, and connections that extend from the original data (Hannula, 2003). These inner monologues can help shed light on the students’ emotional disposition, attitudes and beliefs about mathematics.

Teachers’ interpretations

What happens if we ask a teacher to write what they think are the inner monologues occurring during students’ interactions? Literature review reveals two contrasting scenarios: on one hand, Zan (2007) argues that teachers, when talking about classroom phenomena, speak to their own experience of students. Hence, the most
significant information we can gain from teachers’ comments about students’ interactions is an interesting insight on teachers’ past experiences. On the other hand, Mason’s (2011) discipline of noticing claims that teachers’ interpretations are not naive: teachers' noticing is a relevant part of the profession. Hence, teacher’s noticing is all but personal.

Zan’s (2007) standpoint encourages us to consider teachers’ attitudes, emotions, beliefs that play a role in their fictional writings. Hence, fictional writing can help unfold a teacher’s affect. Of course, the teachers are not passive in the act of reading the codified data: they give meaning to the data, and as a process of active meaning making this is emotionally and cognitively charged.

Mason’s standpoint opens the possibility to go beyond personal experiences and to see teachers as professionals that examine certain phenomena from a privileged point of view: the point of view of those who deal with groupwork activities in their ordinary practice, and professional experience.

**METHODOLOGY**

The methodology reflects this 2-level structure of the paper: we first introduce our interpretations of two examples, which are considered as “data”, then we present the task given to teachers, and our methodology to analyses their responses.

**Data**

The first coded transcript regards a 45 second video clip of a group of four students working on a problem, inspired by Iversen and Nilsson (2005). The problem is:

> A robot walks along a corridor, it turns right with probability 1/3 and it turns left with probability 2/3. The map shows the labyrinth where the robot has to move. Compute the probability for the robot to be in each of the rooms.

Three students, Luca (L), Davide (D), and Marco (M) were selected to be videotaped. We also introduce a new interlocutor to the interaction – the paper (P) with the problem on it, which holds the gazes of the participants at different times of the conversation. We first present the data codified according to our framework (which correspond to the task given to the teachers). In the verbal transcript, we see that both L (00:11) and M (00:37) come to notice that the highest probability is related to the first room, in line with the original formulation of the task by Iversen and Nilsson (2005). The interactive flowchart shows that M is contributing the most proactive statements (n=7) as opposed to L (n=3) or D (n=0). M and D respond to the most proactive statements (each n=5) as
compared to L (n=1 not counting the self-talk as a reaction). There is a marked difference in the number of proactive statements that each person makes that are reacted to – M (n=6), D (n=3), and L (n=1, not counting the self-talk). The gaze arrows show that D never looks at L. D doesn’t look at anyone – he only looks at P when he is speaking. M, on the other hand, spends more time looking at L (n=6) than at P (n=5). At 00:25 D is asking a question while gazing at P. But M is not looking at D – he is looking at L. Then, while M responds to D’s question at 00:27 he continues to look at L. This happens again at 00:34. At the same time L only looks at M three times. Once at 00:15, then again at 00:25 while D is asking a question, and finally 00:36 while M is looking at P.

Why is M so intent on L and why is L ignoring M? We can see something interesting happening at 00:25. While D is asking the question, L and M are looking at each other. But these are not looks of equal intensity. M is clearly more intense in his gaze upon L, who, after a while, glances away from M. From that moment on M continues to be very intensely focused on L. L seems to sense this and diverts his gaze from M, only looking back at him while M is looking at the paper (00:36). Clearly there is an affective aspect to the interaction between L and M. There are emotions, efficacy, will, and motivation in how L and M are interacting with each other. True, all the students express their will to solve the task: D’s questions aimed at letting him follow M’s reasoning, his posture, his repeated and attentive gazes at the paper speak to D’s will to be part, to contribute to the solution.
Table 1: Interactive Flowchart with Gaze Arrows

On the side of both M and D there are many attempts to make their interlocutors act, think or feel (Sfard, 2001). M addresses mostly L, D prompts M. M is working with fractions, he is interested in the procedure. L, instead, seems more interested in understanding the overall sense of the activity (“Why don’t we first compute how many probabilities are there in all?” 00:36). There is a tension between L and M, between conceptual and operational. Moreover, we see that each of these stances prevents each student from seeing the other’s point of view.
The second coded transcript refers to a task regarding the Italian lotto game. In the lotto game, a player bets on 5 numbers, selected from 1 to 90 (with no repetition). In 10 different Italian cities, 5 numbers are randomly and independently extracted from the set of 90 numbers (from 1 to 90). The extracted number is not put again with the others, so that there's no repetition of numbers in the same city. An example of extraction is presented in figure 1. Mathematically speaking, a “cinquina” is a combination without repetition of five different numbers selected from 1 to 90. The player wins if he guesses all the five numbers, or at least two of them.

<table>
<thead>
<tr>
<th>Time</th>
<th>F:</th>
<th>A:</th>
<th>P:</th>
</tr>
</thead>
<tbody>
<tr>
<td>00:16</td>
<td>&quot;...all the cinquinas one can bet on.&quot;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>00:17</td>
<td>A:</td>
<td>&quot;A city on 90 numbers, a cinquina what is made of?&quot;</td>
<td></td>
</tr>
<tr>
<td>00:20</td>
<td>F:</td>
<td>&quot;(speaks over M)&quot;</td>
<td></td>
</tr>
<tr>
<td>00:23</td>
<td>F:</td>
<td>&quot;It is not divided by 5, w..&quot;</td>
<td></td>
</tr>
<tr>
<td>00:24</td>
<td>A:</td>
<td>&quot;(interrupting F) YES!&quot;</td>
<td></td>
</tr>
<tr>
<td>00:26</td>
<td>F:</td>
<td>&quot;(laughing) No, no I trust you, really.&quot;</td>
<td></td>
</tr>
<tr>
<td>00:28</td>
<td>A:</td>
<td>&quot;Well,&quot;</td>
<td></td>
</tr>
<tr>
<td>00:30</td>
<td>F:</td>
<td>&quot;Listen, all the cinquinas one can bet on&quot;</td>
<td></td>
</tr>
<tr>
<td>00:32</td>
<td>A:</td>
<td>&quot;If there are 90 numbers, a cinquina what is made of?&quot;</td>
<td></td>
</tr>
<tr>
<td>00:33</td>
<td>F:</td>
<td>&quot;5 numbers (laughs)&quot;</td>
<td></td>
</tr>
<tr>
<td>00:37</td>
<td>F:</td>
<td>&quot;Right on! (F prompts M to go on)&quot;</td>
<td></td>
</tr>
<tr>
<td>00:39</td>
<td>A:</td>
<td>&quot;So 18 is the number of cinquinas?&quot;</td>
<td></td>
</tr>
<tr>
<td>00:56</td>
<td>A:</td>
<td>&quot;Is it possible?&quot;</td>
<td></td>
</tr>
<tr>
<td>01:45</td>
<td>A:</td>
<td>&quot;So it’s ...18....times 90.&quot;</td>
<td></td>
</tr>
<tr>
<td>01:50</td>
<td>A:</td>
<td>&quot;No, wait. It’s 5 ....&quot;</td>
<td></td>
</tr>
<tr>
<td>01:54</td>
<td>F:</td>
<td>&quot;5 times 90&quot;</td>
<td></td>
</tr>
<tr>
<td>01:55</td>
<td>A:</td>
<td>&quot;5 times 90&quot;</td>
<td></td>
</tr>
<tr>
<td>02:00</td>
<td>F:</td>
<td>&quot;[inaudible, she draws sometimes in the air]&quot;</td>
<td></td>
</tr>
<tr>
<td>02:02</td>
<td>A:</td>
<td>&quot;Yes, right. No. 90 times 90 times 90 times 90&quot;</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: The interactive flowchart of Antonio and Francesca.

Two students, Antonio (A) and Francesca (F), have to compute the number of all possible cinquinas that can be extracted in a city. The transcript (table 2) comes after 5 minutes interaction between them. In addition to the paper, we add another focus to the list of gazes: “somewhere else”. In fact, several times this no-where place holds the gazes of the students. In table 2 proactive statements are few, both from A (n=2)
and F (n=2). F contributes the most reactive statements (n=3), but the self-talk of A (n=6) constitute the majority. F reacts to A’s self-talk at 02:00, and in the first part of the episode no proactive statement is reacted to (not counting the non-mathematical statement at 00:26). Gaze arrows reveal that F is looking mostly at the paper (n=3), or at A (n=2) while speaking, whilst A looks at P only once, and he looks somewhere else both when talking (n=7) and being silent (n=2). A looks at F only once, but at a very important moment of the activity: A is recognizing F as able to contribute ideas. From that moment on, reactive statements show up.

<table>
<thead>
<tr>
<th>RUOTA</th>
<th>1° estr.</th>
<th>2° estr.</th>
<th>3° estr.</th>
<th>4° estr.</th>
<th>5° estr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bari</td>
<td>81</td>
<td>43</td>
<td>31</td>
<td>90</td>
<td>56</td>
</tr>
<tr>
<td>Cagliari</td>
<td>35</td>
<td>47</td>
<td>31</td>
<td>67</td>
<td>74</td>
</tr>
<tr>
<td>Firenze</td>
<td>4</td>
<td>64</td>
<td>56</td>
<td>32</td>
<td>16</td>
</tr>
<tr>
<td>Genova</td>
<td>52</td>
<td>17</td>
<td>13</td>
<td>38</td>
<td>89</td>
</tr>
<tr>
<td>Milano</td>
<td>35</td>
<td>3</td>
<td>34</td>
<td>16</td>
<td>28</td>
</tr>
</tbody>
</table>

**Figure 1:** March 6, 2014 extraction (taken from the Italian official website)

**The Teachers’ Interpretations**

The teachers were provided with a package that consisted of the task so that they may, themselves, engage in the activity – thereby gaining insights into some of the strategies and approaches discussed by the students. The package also included the data consisting of a transcript of the interaction, a coding of proactive and reactive utterances, as well as gaze arrows (a legend has been provided, with proactive, reactive and gaze arrows described).

The teachers were invited to look at the coded transcripts of group interactions, and try to figure out what is happening behind the scenes. Teachers were told that in every group interaction the people interacting have inner speech, goals, and agendas. These things cannot be seen directly, but they can be inferred from the visible attributes of the interaction. Our interest in teachers’ inferences about these invisible aspects of the coded group interactions was clarified.

We then asked teachers to present their analysis either as inner speech written into the blank copy of the transcript (provided with the transcript), or as a paragraph where they can tell in a narrative fashion what they thought was going on.

**Method of analysis**

Our initial analysis was a check for agreements and disagreements between our interpretation of the interaction and that of the teachers. We also compare the interpretations given by different teachers, in order to have a deeper understanding of the phenomenon that is taking place in classroom. The primary goal in this research is, to enlist the experiences of teachers to better understand the turbulent
undercurrents of group interactions, and how such undercurrents shape learning and thinking. Our secondary goal is to check the validity and utility of our methodology, both in encoding group interaction, and in generating reflection about such interaction.

PRESENTATION AND ANALYSIS OF RESULTS

In what follows we present an analysis of the fictional writings of our participating teachers on the aforementioned two excerpts.

Teachers' fictional writings of the first excerpt

These teachers did not see the video, only the codified data (table 1). Nathalie writes:

00:01 D: I don't feel comfortable with this.
00:06 M: I need a pen to write my thoughts.
00:07 L: Let's work systematically together.
00:10 M: I get where you're (L) going.

Nathalie sees a sense of discomfort on D's side, as it is also fictionally reported at 00:17: “D: I feel totally irrelevant”. In our analysis, we have seen a sense of avoidance on M’s and L’s side, as well as the will on D's side to be part of the activity, but we missed this important emotional feature of D's contribution: his feeling of being irrelevant. This is in contrast with the sense of self-confidence that M has. He just needs a pen to write down his thoughts: in this there is agreement between Nathalie’s interpretation and ours. L seems to play the role of the cooperative student, but cooperation is an issue for this group of three students. In fact, Nathalie at 00:25 writes: “D: You’ve left me behind. Please help”, and she concludes the fictional inner dialogue in this way:

00:37 D: I want to understand. This is too fast.
00:42 L: I'm going to do the problem my way.
00:45 M: I'm going to ignore D. He doesn’t get it.

In Nathalie's view, D is the student who goes the slowest, but wants to follow the other students' thoughts. We have seen that D prompts M, but Nathalie also sees that at a certain point M and L decide to leave him behind. D truly wants to understand, to be part of the activity, whilst M rushes to solve the task as soon as possible. L shows a change in his intentions, from “Let’s work systematically together” at 00:07 to “I'm
going to do the problem my way" at 00:42. Nathalie notices that M never looks at D, except when asking for a pen, as if the (absence of) gazes at the beginning of the activity anticipates the act of ignoring D later on. Another teacher, John, sees this ignoring at the beginning of the excerpt and at 00:03 writes “M: Why did we get stuck with David on our team?”. Then, later in the activity, he states:

00:25 D: Slow down, catch me up…
00:27 M: Please, David, just be quiet.

And, at 00:42, “L: I’ll just ignore David”.

The two teachers agree, in their fictional writings, that at a certain point M and L start to ignore D, even if D expresses his will to be part of the activity. But are M and L, themselves cooperating? We had come to say that there is tension between L and M, but we did not dig deeper into detail about cooperation. According to Nathalie’s and John’s fictional writing, we are prone to answer that they are cooperating, but other teachers see a conflict between them (that is, more in accordance with our conclusions about tension rather than cooperation). Let’s look at Helene’s fictional writing:

00:06 M: I see how to do it, so just let me.
00:07 L: I have my own idea, I want to try first.
00:10 M: sshhh! I’m trying to think.
00:11 L: You made a mistake – you should have listened to me.
00:13 M: Stop interrupting me!
00:17 D: I wish you guys would let me at least think about it.

Helene recognises that M initially has an intuition about the solution (“I see how to do it” at 00:06), intuition which is affectively supported by a sense of self-confidence. Helene also acknowledges that L has his own idea (00:07. Furthermore, Patrick writes that M takes the initiative and goes on “regardless of the response of the others”. Moreover, “M is reiterating the known probabilities, almost as if he is reassuring himself and his method”. However, he concludes that “M values L’s agreement over D’s. L does not seem to be interested in arriving at the answer
without basic justification/methodology, and thus pays little attention to M”. Also Patrick comes to agree that M values L’s agreement, but L does not value M’s.

**Teachers’ fictional writings of the second excerpt**

In order to validate our methodology, we allow these teachers to look at the video before writing their fictional accounts. Daniela writes:

00:16  F: Maybe I have not understood.
00:17  A: The problem is simple. A wants to convince F.
00:20  F: I am not convinced, but I can't understand.
00:24  A: You should have understood!
00:26  F: I am not convinced
00:28  A: To follow my reasoning is easy.
00:30  F: I take my time to find the words I need to explain my thinking.
00:32  A: I have 90 numbers and I should divide them in cinquinas, namely?
       Reinforcement to his belief.
00:33  F: I am obliged to follow him.

In Daniela’s view, F needs time and she does not agree with A, while A is confident in his reasoning. Linda writes: “A: What a bore!”, but she adds, later:

00:30  F: He always speeds up.
00:32  A: Again, head to tail.

Such inner speech by A reveals a will to have F with him. But does A really want F with him? And does F want to be with A? In both teachers’ inner dialogues, it emerges that F is not convinced and she needs time to get into the activity. At the same time, both teachers are not saying that F wants to be with M, as also Patrizia, in her inner speech, writes:

00: 28  A: I need to reflect on my own.
00:30  F: Me too.

Patrizia sees a need to go slowly on behalf of both F and A, but is F responding to M’s inner speech in Patrizia’s inner dialogue at 00:30? It is as if both of them recognize in each other that each one of them needs to go on his way. The two students do not want to cooperate at this stage. This is expressed also by other teachers, in different manners. Daniela’s inner speech ends with a sense of resignation (“I am obliged to follow him” at 00:33). Maria goes further on, and at 00:26 writes:
“F: I’ll let you have your way, as usual”. This “as usual” expresses a habit rather than a temporary sense of resignation, pointing to an emotional trait that features F and her relationship with her classmate.

Adriana sees a conflict between A and F:

00:23 F: Oh, what are you saying?
00:24 A: I am better, shut up!
00:32 A: I explain it better to you and to myself.
00:33 F: Hurry up, big genius!

Our analysis tells us that the last move of the students’ interaction features a clearer sense of cooperation. How is it captured by the teachers’ fictional writings? Adriana stresses the conflict between the two students and writes:

01:55 F: You do not do anything by yourself.
02:00 A: I repeat, in order to confirm to myself that I am right.
02:02 F: I have understood!

F’s self-confidence emerges from Adriana’s inner speech. Also Patrizia writes:

01:55 F: For sure it is right.
02:00 A: Could it possibly be right?
02:02 F: This is right. Can't you see?

In this inner speech, F now engages in the conflict with greater confidence, even if the question about her will to be with A still remains unanswered. As Additionally, the teachers do not agree about either A’s will to convince F, or his annoyance.

DISCUSSION OF RESULTS

This paper aims both at questioning the validity of a methodological framework that codifies students’ interactions, and to understand the group interactions in a deeper way, taking into account the interpretations of diverse observers. Our purpose is to check the validity and utility of our methodology, both in encoding group interaction, and in generating reflection about such interaction.

As regards the validity of our methodology, we have presented the coded transcripts of two excerpts and the fictional accounts written by several teachers. In the first case, we watched the video and created codes and inner dialogues. The teachers were able to recreate (and extend) these dialogues from the codes only. This contributes to show reliability of the codes to reflect the video data. In the second case, the teachers saw the video as well, and they were able to create an analysis similar to ours. This contributes to show that our analysis of video is reproducible. Thus, we can say that
this methodology is useful for accessing the turbulent undercurrents of group interactions, and we now discuss and interpret these findings in term of both the mathematical/cognitive issues and the cooperative/social ones that emerge, and their intertwining (speaking to the utility of our methodology in generating reflection).

The teachers’ fictional writing suggests to us that our proposed way of codifying the students’ interactions sheds light on the cooperation between them, their will to cooperate, and their conflicts. The mathematical activity takes place in these interactions, and it is framed by them. For example, in the first excerpt the mathematics that is put forward by M is operational: he computes the probability of being in each one of the rooms. M is solving the task, using fractions and operations between them. D and L, from different stances, express a need to understand the sense of such operations, but M values L’s opinion rather than D’s. At a certain point in the activity, all the teachers acknowledge that M and L discard D. For some teachers, M and L also stop cooperating, and wish to solve the task separately. Why is cooperation so central to the teachers’ attention? We see a social and cultural issue here: in the last decades, much effort has been paid by teacher educators, researchers, governments, etcetera, to promote cooperative learning among students. Teachers pay attention to cooperative aspects of the activity because historically and socially this is today a fundamental issue for their everyday practice.

The teachers’ fictional writing, however, also suggests that cooperation takes place within competition and conflicts. We can read competition as desire to be best, and conflicts refer back to different views of mathematics and different attitudes towards it. The conflict between L and M in the first excerpt is a conflict between two views of mathematics: M’s pragmatic effort to solve the task, and L’s theoretical effort to understand the sense of what is being done. Moreover, desire to keep up speaks of D’s motivation to be the best he can be, likewise M’s discarding of D shows his desire to be best and not to be impeded by another’s slow thinking.

Also in fictional inner dialog for A and F we see competition and conflict, in a different way from the first excerpt. We add that the transcript does not at all reflect what is happening in the video. Looking only at the video, as the teachers also did, opens the possibility to answer to the question about F’s will to be with A. F’s body language seems to indicate that she is definitely “in" the conversation. A’s body language at the beginning tells that he does not want to be in the conversation. A is very convinced to that he is right, to the point that he keeps F away from him and his reasoning. F acts like D in the starting moment: she tries to stop A, she wants to
understand and go slower. But, unlike D's emotional state, no teacher reports that F feels "totally irrelevant".

The first episode ends with a failure in cooperation: the teachers' inner dialogues tell that L discards M, M discards D, and each student work on his own. The second episode begins with a similar lack of cooperation, with A and F discarding each other. Unlike the first episode, however at a certain point the two students start to truly cooperate. This happens after A fails to find the solution by himself, and he recognizes F as an interlocutor. In the first excerpt, M recognizes L as an interlocutor, but L does not recognize M: the conflict remains open between two views of mathematics. In the second excerpt, A allows F the opportunity to explore and understand the task, thence F becomes ready to follow A, and to be a good interlocutor. In the first excerpt, the outcome of the activity is the individual result put forward by M. In the second excerpt, the results comes from A as well, the student with the highest self-confidence in his mathematical ability, but only after an interaction with F, who has been allowed to play her role.

References


Recent studies claim that mathematics teachers without a formal qualification for mathematics negatively impact student achievement. We may ask the question: Why does teacher certification matter? Teaching and learning processes in mathematics classrooms taught by out-of-field teachers have not been examined yet. In this study, we will report on a lesson taught by the out-of-field mathematics teacher O. By analyzing his/her teaching practices by means of the TRU Math Scheme and the concept of subject-related teacher identity, we will demonstrate how O.’s mathematical identity – and not simply mathematical knowledge deficits – leads to teaching practices that prevent a mathematically powerful classroom.

OUT-OF-FIELD TEACHING IN MATHEMATICS

Out-of-field teaching, i.e. teachers teaching a subject or a year level without a formal qualification to do so, is an international phenomenon affecting not only educational systems (Ingersoll, 1996, 1998) and schools as organizations (Steyn & du Plessis, 2007), but also students’ learning and development. The actual requirements which are not met by out-of-field teachers, so that they are not considered as being formally qualified, differ from country to country (Hobbs & Törner, 2014). The German teacher O., whose teaching will be discussed in the following, has neither studied mathematics at university nor did he attain the second state examination in mathematics which usually needs to be passed after a one to two-year post-university period of practical teacher training. Therefore, he does not hold the so-called Lehrbefähigung for mathematics and is not formally qualified to teach the subject (Bosse & Törner, 2014).

Goldhaber and Brewer (2000) claim that teachers who have a formal qualification for teaching mathematics have a significantly positive impact on student achievement. This is also stressed by Hawk, Coble, and Swanson (1985) and by Dee and Cohodes (2008), who report a positive relationship between a certification in mathematics and effective teaching in the actual classroom.

In Germany, these findings can be confirmed especially regarding low-achieving students: Richter, Kuhl, Haag, and Pant (2013) conclude from their studies that these students perform even worse on tests in classes that are taught mathematics by out-of-field teachers.

These inquiries focus on student outcomes and teacher competencies; however, the mediating processes have been neglected and one question has not been clarified:
Why does certification matter? In order to attempt to answer this question, we place the emphasis on the mathematical teaching and learning processes rather than on measuring de-contextualized shortcomings in mathematical knowledge caused by a lack of teacher education. Thus, our talk will be subdivided into three parts.

Firstly, we will describe two theoretical approaches and corresponding methodological considerations. By means of the concept of teacher identity we can uncover hidden variables of specific teaching practices (Wenger, 1998). We will define teacher identity in a holistic manner by referring to a broad spectrum of literature (Beauchamp & Thomas, 2009). Furthermore, the Teaching for Robust Understanding in Mathematics Scheme (TRU Math Scheme) (Schoenfeld, Floden, & the Algebra Teaching Study and Mathematics Assessment Project, 2014a, 2014b, 2014c) provides a five-dimension model of mathematically powerful classrooms. If we want to gain an insight into teaching processes, Schoenfeld’s model can help us structure lessons and identify crucial characteristics of out-of-field teaching which may result in low student achievement.

Secondly, we will investigate out-of-field teacher O.’s case and present the results of the analysis of one of his mathematics lessons.

Thirdly, we will draw some conclusions and present ideas in order to answer the question of why certification matters. Moreover, we will show why it is important to consider teacher identity in a holistic perspective rather than exclusively focusing on knowledge gaps when designing support measures for out-of-field mathematics teachers.

THEORETICAL PERSPECTIVES

Teacher Identity in Practice

In terms of Wenger (1998, p. 149) “there is a profound connection between identity and practice” assuming that identity and practice are “mirror images of each other” (ibid.). In this respect, the way of teaching mathematics as a specific teacher’s practice on the one hand depends on the teacher’s identity; on the other hand, teaching practice shapes the teacher’s identity. The experiences a teacher has gained while teaching and learning in mathematics classrooms during his personal and professional life have an impact on ‘who the teacher is’ when teaching mathematics, to what extent the teacher claims to be a mathematician or not, on what the teacher thinks is important (or not important) when doing mathematics, and so forth.

We argue that teacher identity is not a static entity but a highly dynamic construct which depends on and has an impact on situated practices and their (social, biographical, institutional etc.) contexts (Bosse & Törner, 2013). Therefore, elements of a specific teacher’s teaching can be explained by referring to the teacher’s identity.
Dimensions of Mathematically Powerful Classrooms (TRU Math Scheme)

The TRU Math Scheme is an analytic framework for characterizing five relevant, minimally overlapping dimensions of mathematics classrooms (Schoenfeld et al., 2014a, p. 2) which are defined on the basis of well-grounded theories and empirical evidence (Schoenfeld et al., 2014a, pp. 6–23):

1) The Dimension of The Mathematics, i.e. the extent to which the mathematics discussed is focused and coherent, and the extent to which connections between procedures, concepts and contexts are addressed and explained.

2) The Dimension of Cognitive Demand, i.e. the extent to which classroom interactions lead to an environment of productive intellectual challenge which is helpful for the students’ mathematical development.

3) The Dimension of Access to Mathematical Content, i.e. the extent to which classroom activities support the engagement of all the students.

4) The Dimension of Agency, Authority and Identity, i.e. the extent to which students have opportunities to explain, to put forward mathematical arguments and to build on other students’ ideas, in a way that strengthens their capacity and their willingness to engage themselves in mathematics, resulting in positive mathematical identities.

5) The Dimension of Uses of Assessment, i.e. the extent to which teachers analyze the students’ thinking and integrate related responses into instruction and the extent to which basic conceptions are developed as well as to which misconceptions are recognized and dismantled.

Having in mind that these five dimensions can explain why a classroom produces (or does not produce) powerful mathematical thinkers, the scheme is helpful for our purposes. Furthermore, the scheme provides a scoring rubric for capturing the domains’ presence in instruction. For this MAVI-paper with limited scope, the analysis will be reduced to domains 1) and 4).

METHODOLOGY

General Remarks

Teacher O. is the 15th interviewee within a sample of an authors’ dissertation project. The sample is arranged in alphabetical order. He/she was selected for the analysis because he/she was the only teacher who allowed the authors to visit his/her classroom when the paper was written. This underlines the methodological challenges of investigating out-of-field teachers: They seem to be afraid of being judged and having to admit not having performed optimally. Teacher O. is formally qualified for teaching theology and sports; however, he/she has been teaching mathematics for 30 years. He/she does not have any certifications related to mathematics and has never attended a teacher training course with a mathematical focus.
As a methodological consequence of the case-study approach, we cannot generalize the study’s findings, i.e. we cannot make statements about the practice of out-of-field teaching in mathematics classrooms per se. Nevertheless, we will be able to clarify whether classroom practices are only based on the teacher’s knowledge or whether they can also depend on identity-related realms. In addition, we will characterize how the interdependence between identity and practice functions.

**Data Gathering**

The mathematics lesson of teacher O. lasted 45 minutes and was filmed at a German comprehensive school (lower secondary school) in the federal state of North Rhine-Westphalia in February 2014. The mathematics class was attended by 15 or 16 year old students and is called *G-Kurs*. This means that the teenagers only learn the basics in mathematics at the lowest level that is possible within the German educational system. The stationary camera was placed in the back left corner of the classroom.

The class had been selected by teacher O. and not by the authors. O. did not explain his choice and only offered the authors to visit the respective lesson that will be analyzed in this paper.

Teacher O. had been interviewed one week before his lesson was filmed. The qualitative, semi-structured interview lasted 50 minutes and was conducted by means of an interview guideline. The respective guideline contained questions referring to the subject-related teacher identity and thus relating to a broad field of cognitive and affective-motivational aspects in terms of Grootenboer and Zevenbergen (2008): What does good teaching in mathematics classrooms mean for you personally? What exactly is mathematics for you? How did you think about mathematics and mathematics classrooms during your own schooldays? What is important for being a good mathematics teacher?

By asking such questions, we can provide a holistic perspective for examining the teacher’s identity (Bosse & Törner, 2013). This offers the advantage of being able to identify connections between professional realms concerning mathematics and teaching mathematics: Those connections that are not related to mathematics and those that are not professional at all because they concern the personal sub-identity. Especially integrating non-mathematical and non-professional domains is helpful for understanding out-of-field mathematics teachers’ teaching practices, since we can assume that such teachers usually have never been engaged with mathematics beyond school.

**Data Analysis**

The lesson is structured by the types of activities in terms of the TRU Math Scheme (Schoenfeld et al., 2014c). Thus, we can define different sections of the lesson. For each section, the content is outlined and the TRU Math Scheme Scoring Rubric is used (ibid.). A scoring of 3 indicates the best fulfillment of the dimension that is
possible, contributing to a mathematically powerful classroom. A scoring of 1 indicates that the classroom is least able to produce powerful mathematical thinkers. The scoring is done by identifying the qualitative characteristics of the lesson and then comparing them to those suggested by the scoring rubric. This will be done for both dimensions 1) and 4).

The interview with teacher O. has been transcribed. In order to understand his/her teaching practices and the results of the scoring, the observed teaching practices will be compared to aspects of O.’s teacher identity named in the interview.

RESULTS AND DISCUSSION

In the following, we will first give an overview of the filmed lesson categorized by the activities. Secondly, we will present the results of the analysis gained with the help of the TRU Math Scheme Scoring Rubric by assigning each activity a score from 1 (poor) to 3 (strong). Thirdly, the results referring to dimensions 1) and 4) will be explained and discussed. Fourthly, we will relate the observed teaching practices to the qualitative data gathered during the interviews. Finally we will present an actual incident taken from the classroom in order to show and discuss the consequences of the findings directly in the data.

Lesson Overview

The structure, procedure, activities and content of the filmed lesson are set out in table 1.

<table>
<thead>
<tr>
<th>Time [min:sec]</th>
<th>Activities</th>
<th>Content</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1 00:00 – 04:30</td>
<td>L</td>
<td>organizational</td>
</tr>
<tr>
<td>#2 04:30 – 12:00</td>
<td>W</td>
<td>Checking homework: Students were supposed to solve combinatorial tasks (urn model with 4 balls; 2 balls are drawn one after another; balls are each placed back in urn after drawing).</td>
</tr>
<tr>
<td>#3 12:00 – 13:30</td>
<td>W</td>
<td>organizational</td>
</tr>
<tr>
<td>#4 13:30 – 17:45</td>
<td>W</td>
<td>O. gives a similar task (urn model with balls placed back in urn; 3 balls are drawn one after another): 3 white balls, 2 red balls and 1 blue ball are in an urn. O. asks for “combinations” and for the related probabilities. Work is done at the board.</td>
</tr>
<tr>
<td>#5 17:45 – 21:00</td>
<td>W</td>
<td>New content: Same task as in #4, but without</td>
</tr>
</tbody>
</table>

1 Activities are labelled in terms of the TRU Math Scheme: L=Launch; W=Whole Class Discussion; I=Individual Work
placing the balls back in the urn.

#6 21:00 – 22:00 W Short discussion about the difference between #4 and #5.

#7 22:00 – 38:15 I Students copy the writing from the board and are supposed to solve tasks in their school book similar to the task in #5 on their own.

#8 38:15 – 38:55 W Students have a problem understanding an assigned problem concerning a tree diagram and many students ask teacher O. for help.

#9 38:55 – End W The corresponding task is solved on the blackboard – without a tree diagram.

Table 1: Structure, procedure, activities and content of the filmed lesson

The label “W” (Whole Class Discussion) is most suitable for describing the activities in terms of the TRU Math Scheme. However, the ‘discussions’ during the lesson which will be analyzed are very short dialogues between the teacher and single students. There is no real conversation engaging many students. The lesson is characterized by one-directional call-and-response communication and not even “language games” (Bauersfeld, 1995) are ‘played’.

TRU Math Scheme Scoring Rubric Results

An overview of the scoring results is given in table 2.

<table>
<thead>
<tr>
<th>Activities</th>
<th>Dimension 1 (Mathematics) Scoring Results</th>
<th>Dimension 4 (Agency, Authority and Identity) Scoring Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1 L</td>
<td>N/A²</td>
<td>N/A</td>
</tr>
<tr>
<td>#2 W</td>
<td>1 out of 3</td>
<td>1 out of 3</td>
</tr>
<tr>
<td>#3 W</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>#4 W</td>
<td>1 out of 3</td>
<td>1 out of 3</td>
</tr>
<tr>
<td>#5 W</td>
<td>1 out of 3</td>
<td>1 out of 3</td>
</tr>
<tr>
<td>#6 W</td>
<td>1 out of 3</td>
<td>2 out of 3</td>
</tr>
<tr>
<td>#7 I</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>#8 W</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>#9 W</td>
<td>1 out of 3</td>
<td>1 out of 3</td>
</tr>
</tbody>
</table>

² The label N/A is used if the activity is mostly for clarifying organizational issues. Concerning #7, the authors of the TRU Math Scheme recommend not using data of individual work if a stationary camera is used.
Table 2: TRU Math Scheme Scoring Rubric Results concerning dimensions 1 and 4

**Dimension 1: The Mathematics**

Table 2 shows that the way the mathematics is discussed in every section of the lesson does not contribute to a mathematically powerful classroom.

The urn model is used for visualizing the procedure of drawing only (see #4 and #5); teacher O. does not explicitly connect the model to his calculations of probabilities – or makes the students do so. This becomes vital when the teacher asks the students about the difference between returning the drawn balls to the urn or not returning them to the urn (see #6). The students exclusively refer to the calculation, i.e. the changing denominators of the fractions. They do not give reference to the urn model or to the changing probabilities for drawing a ball of a specific color.

After the students have characterized this difference by referring to the changing denominators, not the students but the teacher connects the arithmetic changes to the variation of the urn model. Further, O. does not let the students discover that the denominator is reduced by 1 after every draw. He/she just ‘comments on’ this mathematical structure after a student has (non-specifically) remarked that the denominators change.

Furthermore, the teacher uses the term *combinations* incorrectly as he/she constantly considers the sequence of the balls drawn. This means that O. actually asks for *permutations*, e.g. ordered samples, but O. does not mention whether the sequence actually plays a role when defining probabilities or not. He/she just writes – formally careless – equations such as “RRB = 2/6 * 2/6 * 1/6 = 4/216” on the board.

O. does not embed the mathematical problem into a real context. He/she doesn’t put any emphasis on the urn as a *model*. As a result, students perceive the urn model as the actual context.

There is no real problem solving or reasoning, because the teacher only asks the students to calculate the probabilities. As referred to above, the urn model and aspects of problem solving related to it are only integrated by the teacher and not by the students themselves.

**Dimension 4: Agency, Authority and Identity**

O.’s dominant role in the solving process directly influences the scoring results in dimension 4). The visualization of the solution is done entirely by the teacher and the urn model (sketched on the board) is exclusively used by O. as an instrument for solving the problem. Additionally, O. is the only one who uses the sketched urn with balls as a tool for visualizing the drawing procedure in #5 by clearing or adding the letters “W”, ”R” and ”B”.

The teacher almost exclusively tries to initiate conversations. The students’ response turns are short; they mainly concern the calculated results and they are constrained by what the teacher says. The students have almost no chance to explain their solutions.
because O. mainly asks closed questions. Moreover, the students’ speech is confined to procedures and results, not to their thinking or reasoning. Reasoning and explaining is done by the teacher. In the event that a student answers a question incorrectly, O. explains why it is wrong or why the result cannot be right. Fundamentally, students do not respond to and build on each other’s speech.

The modeling of ‘drawing’ and focusing on corresponding stages is entirely done by the teacher. In terms of communication, the students simply ‘fill in’ the probabilities as calculated results. O. primarily asks the students to calculate (“Calculator!” (15:18) or “Calculate!” (18:51)) and – except for #6 – not to reason and explain.

Teacher Identity in Practice

The interview clarifies that O. states to enjoy mathematics very much and had much fun doing mathematics when he/she was at school himself/herself. Besides, he/she believes that he/she is mathematically talented. Since he/she has not studied mathematics, he/she mentions some deficits regarding his/her knowledge in the beginning of teaching this subject. But today, after having taught mathematics “for so many years” (Teacher O., personal communication, January 27, 2014), he/she thinks that he/she is – “naturally” (ibid.) –able to do mathematics.

O. explains that he/she liked doing algebra and everything “where one can calculate something” (ibid.) in his/her schooldays. He/she still likes this much more than “doing other mathematical things” (ibid.).

His/her primary motivation for teaching mathematics is that he/she enjoys calculating. Moreover, mathematics is in his/her view a subject which is much clearer than other subjects. O. thinks that mathematics is also less arbitrary or user-defined than geography or social studies. He/she points out that a teacher can proceed as defined by the school book when teaching mathematics. Therefore, O. explains that he/she works with the book or quotes mathematical tasks from it in every single lesson. In contrast to the other subjects he/she teaches, he/she regards the school book as the dominant medium in his/her mathematics classrooms.

O. thinks that mathematical classrooms are powerful, if student achievement is high. In O.’s view, the mathematics teacher’s task, in order to contribute to a mathematically powerful classroom, is to motivate the students to participate in mathematics.

Bosse: Is there something like mathematical talent?
O.: Yes […] I can see students who are already able to see connections, some who are able to acquire this and some who are like a dying duck in a thunderstorm only sometimes uttering: “Ah – now I understand.” However, they haven’t actually seen the right thing […]. The insights they have gained are correct regarding a specific case but they are not true in general. Therefore, they do not recognize structures; they do at most understand an example. […] And many of the students in my class do not recognize structures.
Bosse: What do you think about your skills of promoting students’ conceptual thinking? Or is this something you struggle with?

O.: I would say: I could need support; but this is true for every mathematics teacher. This is probably the most difficult thing to do.

Bosse: How do you activate your students’ thinking?

O.: How do you? How can you do this for example? […] I have no idea.

O. does not know how he/she can support the students in order for them to achieve the goal of being able to focus on central mathematical ideas and building corresponding mental constructs of mathematical conceptions. Speaking in terms of the TRU Math Scheme, he/she believes that students either have a talent for understanding and explaining connections between procedures, concepts and contexts, or they do not.

**A Consequence: Students’ Difficulties with Tree Diagrams**

During the lesson, the students do not know how to solve a combinatorial problem by means of a tree diagram when working on their own (see #8):

**Student:** It’s so complicated. I don’t understand the thing with the tree. I cannot imagine anything with this. But I can calculate in the same way as we did on the board.

**O.:** The tree seems to be more difficult than calculating.

**Students:** Yes!

**O.:** Ok. Let’s do it like this: First, we calculate and then we try the tree – because you have to handle the tree [on the final test], too.

[…]

**Student:** I hate the tree!

In this scene, O.’s preference for calculating *and* its impact on the students’ learning become evident. The tree is not seen as a tool for broadening conceptual knowledge of probability in multi-stage processes. The teacher’s only reason for dealing with tree diagrams is that they will be tested on the final exam.

**CONCLUSIONS**

We have just seen that O.’s early experiences as a learner of mathematics which partially construct his/her mathematical identity still have an impact on the practices which have been observed in the mathematics lesson he/she is involved in as a teacher. His/her preference for *calculating* determines what he/she thinks is important when solving combinatorial problems related to urn models. He/she is not aware of the fact that he/she is also responsible for promoting students’ conceptual knowledge and understanding, but thinks that acquiring mathematical concepts is something that is determined by natural talent.
This has consequences for the quality of his/her teaching. For this study, implications for the TRU Math Dimensions 1) and 4) could be confirmed. Mathematics is reduced to aspects of calculation for almost the entire lesson and the students’ role in this classroom is to make these calculations. The students’ “identities as doers of mathematics” (Schoenfeld et al., 2014c, p. 1) are ignored and students are not given the “opportunities to explain, make mathematical arguments, and build on one another’s ideas” (ibid.). Regarding these results, we can assume that O.’s students will not develop their mathematical agency; in our opinion, the only doer of mathematics in the analyzed lesson – and therefore the author of mathematical practices – has been teacher O.

Furthermore, it is problematic for the students’ learning if a teacher emphasizes procedures and calculating because of his own preferences and assumes that the acquisition of conceptual understanding is primarily a matter of gift. In such a case, only those students being able to build conceptual knowledge on their own will achieve highly. This may be one reason for the results of the studies referred to in the introduction.

If we want to support out-of-field teachers like teacher O., it might not be enough to look at deficits concerning mathematical content knowledge. This is especially the case if a teacher is very good in school mathematics and likes solving mathematical problems which allow focusing on procedures and calculating. Developing O.’s mathematical identity is at least as important as overcoming mathematical shortcomings. As an example: Initiating the insight that mathematical procedures can only be learned on a long term basis if they are being interwoven with fruitful mathematical conceptions – and their acquisition is not the result of an inherent mathematical talent.

The case of teacher O. also reveals that even an experienced out-of-field mathematics teacher refers to personal facets of identity as one strategy of practice if the professional knowledge is in deficit. Instead of referring to helpful, professional teaching practices of mathematics instruction, which are fruitful for nurturing conceptual knowledge, the teacher’s own preference for calculating is essential when selecting and prioritizing resources for teaching mathematics. As a consequence (see also Schoenfeld, 2011), O.’s goal is to teach the procedures of calculating probabilities instead of focusing on the construction of conceptual knowledge of probabilities in multi-stage processes.

References


WHAT IS A “GOOD” MASSIVE OPEN ONLINE COURSE?

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Department of Mathematics, Polytechnic of Milan (Italy)

This paper attempts to apply an affective lens to analyse reports and guidelines of Massive Open Online Courses (MOOCs) developed at several universities around the world. Specifically, we outline a panorama of definitions of “quality” that emerge from official reports, guidelines and personal interviews. We select some MOOCs that represent a significant portion of the present state of the art and we apply an interpretative approach in order to understand the beliefs of MOOC producers.

INTRODUCTION AND LITERATURE REVIEW

Massive Open Online Courses (MOOC) represent a development of e-learning with the aid of the newest ICT resources. Ferrari (2011) observes that MOOCs contribute to the exploration of the potentialities of new ICT resources at three levels: cognitive (the students became conversant with the ideas and the methods that are specific for the subject matter), meta-cognitive (in particular, self-assessment and ability to portray one’s own curriculum) and affective (beliefs, emotions, attitudes, which drive the students’ choices and actions). Ferrari (2011) underlines that MOOCs themselves become affectively charged, and contribute to shape emotions and attitudes that are crucial for the learning processes. However, researches in Mathematics Education focusing on the potentialities of MOOCs with respect to either affective, meta-cognitive or cognitive levels are few (Ferrari, 2011). Thus, we would like to investigate affective issues that MOOC producers have encountered. MOOC producers, which have been scarcely investigated, represent a third party other than teachers and students. MOOC producers are not necessarily teachers or researchers in Mathematics Education.

Following Kontorovich (2013), who analysed problem posers in international mathematical competitions, we are interested in analysing the MOOC producers’ goals, to point out the characterising features of “good” online courses. Specifically, when talking about MOOCs, one may wonder about their efficacy, their quality and their innovative features with respect to more traditional courses. As Kontorovich (2013) has also observed with respect to mathematical problem posers, experts make use of descriptors such as “beautiful” or “attractive”, without fully enabling us to access the meaning beyond these descriptors. Thus, we are interested in more deeply analysing the meaning that adjectives such as “effective”, “innovative” and “good” have for some MOOC producers.

THEORETICAL BACKGROUND

As a matter of fact, MOOC producers possess their own beliefs, attitudes, knowledge and emotions towards teaching, learning, themselves, etc. Within research on mathematical beliefs, McLeod (1992) had identified four main objects of beliefs:
Brunetto, Andrà

mathematics as a discipline, self, mathematics teaching (and learning), and social context. As concerns the origin of beliefs, McLeod (1992) also considers both social context (culture) and individual experiences to contribute to their formation. We focus here on beliefs related to the social context of MOOC and MOOC producers. By “context” we mean specifically the political, cultural, social bounds within which MOOCs have been developed.

Hannula (2011) observes that much research in the affective field adopt a framework (inspired by McLeod’s work), where emotions, attitudes and beliefs are located on a continuum: beliefs are seen to be the most cognitive and stable, emotions as the least cognitive and stable, and attitudes in the middle for both dimensions. Prior research regarding stability has pointed out an issue that is also relevant for our research: namely resistance to change. In our research, we see resistance to change, for example, in terms of willingness to look for indicators of quality, efficacy and innovation other than the established standards. True, resistance to change is bound to the context. We will attempt to identify the factors that may influence this change.

Furthermore, MOOC producers are not isolated, but (a) they usually work in teams, and (b) they have to confront—sometimes to contrast—the products of their work with many stakeholders, experts of other disciplines, feedback from students, etc., MOOC producers compare their work with other MOOCs which are generally thought to be “successful”. We wonder if such a comparison can be a factor that drives a change in their beliefs.

**METHODOLOGY OF RESEARCH**

**Type of data**

If one searches for data regarding the quality of a specific MOOC, one can often easily find the data log. We can say that the data log (i.e., number of accesses, enrolment figures, number of times a video has been viewed, countries of provenience) are taken by MOOC producers as indicators of quality. In a sense, this constitutes the established standards for MOOCs, but we are interested in going into more detail in order to better understand the MOOC producers’ beliefs about quality. To this end, we collected reports on MOOCs that cover a significant portion of the present “state of the art” worldwide; we analysed data not only from concise and marketing-oriented online reports about some MOOCs, but also extensive official guidelines, posts and articles online, internal reports and notes from personal interviews.

We analysed many documents in order to determine common features, to outline similarities and differences, and to compare them. A first dichotomy that characterizes our study is between the private and the public dimensions. For example: emails, meetings, chatting, and “rumors” are private, whilst reports, guidelines, interviews and brochures are public. We also conjecture that to a certain extent the private contains the public. In fact, in the private sphere the producers
discuss and decide how to design the MOOC, what they want to focus on, which kind of tools they want to use, etc. The product of such private interactions are: guidelines, reports, and infographics. This conjecture has been tested and constitutes the first phase of our analysis. Both private and public information are not raw data, but reports of other raw data (number of enrolments, for example). This is an interesting feature of these data. Meta analysis is a discipline that considers reports as data (not as analysis), and triggers the understanding of a phenomenon by looking at these results as data themselves.

Meta-analysis, indeed, can help us gain more precise knowledge on the overall effects concerning a phenomenon, given that different studies are usually done with different standards, different definitions and methods, different populations. Meta-analysis searches for patterns, similarities and agreement within independent reports on the same phenomenon. Usually, in quantitative studies, a homogeneity test is performed in order to determine if the effects measured by the studies are sufficiently similar (Viechtbauer, 2007). Additionally, in qualitative studies (which is our case), an analogous investigation can be performed. In fact, even without a quantitative apparatus that may give a numerical estimate of the aforementioned measures, a researcher can be interested in a common effect as it emerges from independent studies on different populations, and he/she may be concerned with variation both within studies and between studies, searching for a “satisfactory” degree of homogeneity. In a qualitative study one usually does not have a mean, but some trends, common behavior, observations that lead one to consistently claim that a certain model is reliable.

**Method of analysis**

In this research, we investigate what the MOOC producers mean by quality. At first glance, the information provided by MOOC producers reveal that the common standard upon which the producers focus, is the data log. However, we need to carefully search for other indicators of “quality” and we decided to use the words that form the MOOC acronym, Massive, Open, Online and Courses, as variables. In order to interpret the data in terms of these words, we need to introduce indicators:

- The “**massive**” feature is taken as the number of potential users. When English is used, we consider the MOOC as massive; in the case of a different language, we also consider scalability and community target. Scalability takes into account the spread of language used to produce a MOOC. With regard to the target community, a MOOC designed for a small community is massive if all the members of that small community enroll in the course. In this sense, the target community has been reached in a massive way.

- With regard to the term “**open**”, we consider 3 possibilities: (a) access is free, (b) a fee is necessary in order to enroll as a student, or (c) a fee is necessary, not to attend the course, but to obtain a certificate as a graduate.
Moreover, we add two aspects to this feature: ease of access and engagement of the student.

- The “online” term identifies which type of platforms are used and which kind of interactions are available. We also distinguish between MOOCs in which the video lectures are synchronous with the students attending them, and MOOCs in which the lectures can be accessed at any time.
- The “course” term identifies the awareness of didactical, pedagogical and educational issues related to a MOOC. In fact, MOOCs can be seen as (a new way of ) teaching. Moreover, different kinds of feedback can be exploited by MOOC developers. We, thus, attempt to determine if MOOC developers see themselves as “teachers” and if they take educational issues into account when framing the quality of the course.

We also identify other two variables regarding two crucial aspects: (1) “video”, which takes into account the possibility of mentioning information regarding video format, and (2) “brand”, which takes into account the attention paid to brand, visibility and prestige, for the institution developing a MOOC. Table 1 summarizes the 6 variables and the indicators we have outlined.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Indicators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Massive</td>
<td>Number, language, scalability and community target.</td>
</tr>
<tr>
<td>Open</td>
<td>Free/fee to enroll/to receive a certificate, ease of access and engagement.</td>
</tr>
<tr>
<td>Online</td>
<td>Platform, interaction, (a)synchronous and 24/7 access.</td>
</tr>
<tr>
<td>Course</td>
<td>Didactical, pedagogical and educational issues, feedback</td>
</tr>
<tr>
<td>Video</td>
<td>Format, length and video quality.</td>
</tr>
<tr>
<td>Brand</td>
<td>Brand, visibility and prestige.</td>
</tr>
</tbody>
</table>

Table 1: Variables and their indicators for our investigation on quality.

These indicators mirror the common reflection in the community MOOC developers, but can assume different meanings if one of them is combined with a subset of the others. Namely, different views on MOOC quality can be gained by looking specifically at the indicators and their relationship. For example, a belief system that considers “massive” in terms of “target community”, and “open” as “engagement”, is different from a belief system that considers “massive” as “number” and “open” as “ease of access”.

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Data collection

We collected 23 sets of data, from different MOOCs. The public data consist of: nine infographics from different institutions (Melbourne University, Miriada X, edX, Coursera, FUN), one blog post by Justin Reich (Harvard University), two blog posts by Matthieu Cisel (Normal'Sup Cachan), one interview of Sebastian Thurn (Udacity), one paper about video format (Guo, 2014), one report and one set of guidelines by FUN, one report by MIT as well one report from Coloumbia University. The private data are comprised of: one internal presentation from EPFL, a private interview regarding a Zurich MOOC, two internal reports from a Milan MOOC, several emails between producers in Milan, and one private interview regarding a Milan MOOC. We analyzed the documents to see: (i) whether the variables are mentioned in each instance; (ii) which indicators are used by MOOC producers. Since it is not always possible to collect both public and private information for each MOOC, we were only able to gather a partial information. Therefore, to investigate (ii), we choose to examine the MOOCs which have the largest number of heterogeneous documents. The five different MOOCs we have selected are representative of the general situation: (1) the MOOC developed at the University of Melbourne (report 2012), (2) the national French MOOC developed at Français Université Numérique (guidelines 2012, report 2013), (3) a course developed at the University of Zurich (notes from interviews), (4) the MOOC developed at Columbia University (report 2014), and (5) the first Italian MOOC developed at the Polytechnic of Milan (notes from interviews). The Italian MOOC is the most recent in this list, since it was launched at the beginning of June, 2014.

ANALYSIS AND DISCUSSION OF DATA

(i) Are the variables mentioned in each type of information?

To answer to this question, 23 documents are considered. Table 1 summarises the different types of data we got (6 private information, 4 reports, 4 blog posts/web article, 9 infographics) and the variable that are mentioned. The number in each cell indicates the number of documents that mention the indicator of the variables.

<table>
<thead>
<tr>
<th></th>
<th>Massive</th>
<th>Open</th>
<th>Online</th>
<th>Course</th>
<th>Video</th>
<th>Brand</th>
</tr>
</thead>
<tbody>
<tr>
<td>private info (6)</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>reports/guidelines (4)</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>blog posts/web articles (4)</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>infographics (9)</td>
<td>9</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 2: summary of the type of information and the mentioned indicators for each one of the 23 documents analysed in the first phase of the research.
From Table 1 we see that the answer to question (i) is negative. Furthermore, we can see that there are many common features for each type of information: for example, the private information consider aspects linked with all of the variables, with specific attention given to the didactical aspects (“Course” is present in 6 out of 6 cases), and a high presence of video and openness issues. Infographics focus mostly on the “massive” variable, and rarely contain the other variables. Therefore, we note that the private dimension is the best in order to understand the producers’ attitude regarding MOOCs, even though it is necessary to look at infographics in order to fully understand the meaning of the word “massive”. Moreover, our starting conjecture that the private data contains the public information is partially true, since information about massive issues can be found in infographics rather than in private information. This fact means that we have to consider all kinds of information in order to understand MOOCs producers’ attitudes towards quality.

(ii) How do the MOOC producers interpret the variables? Which kind of relationship exists between the variables?

From Columbia University we have the report MOOCs: Expectations and reality (Columbia University, report 2014), its table of contents is shown in Figure 1, and allows us to grasp the priorities of this MOOC’s developers: the first issue addressed is “Extending Reach and Access”, the second issue is “Building and Maintaining Brand”, followed by “Improving Economics: Reducing Costs or Increasing Revenues”. We can observe that the first three issues do not regard aspects of teaching and learning, which are the focus of the last three issues (among six). By only looking at the table of contents, we infer that the first aim is the prestige (“brand”) of the University in terms of ease of access (“open”) and visibility (“brand”). Another important feature
of this table of contents is that economic aspects (“brand”) come before the didactical aspects are understood as improving, innovation and research about teaching methods (“course”) in Figure 1. By also reading the content of the report, we can say that “good” and “effective” MOOCs are, for these producers, MOOCs that can be accessed easily (“open”) to the point that producing a MOOC is seen as “mission to make education universally available” (p.53), MOOCs which are well known, and MOOCs that contribute to the economic improvement (“brand”) of the University.

In the infographic produced by Melbourne (see top left of Figure 2), didactical issues lie in the background. In the foreground are, for example, the number of answered quizzes or submitted surveys (“massive”) and the average time students spend on the platform (“massive”). In Figure 2 we read that 348220 students enrolled at the MOOC. The number of submitted surveys is 10542, and the number of submitted “peer assignments” is 9969. Our interpretation of such a practice is that it helps the MOOC producers to focus attention on “big numbers” (“massive”). Computing a simple percentage, we can see that only 3% of enrolled students submitted a survey, and 2.8% of them submitted a “peer assignment” (“course”). Hence, even when looking at more detailed data, “quality” for MOOC producers is read in terms of integer numbers that frame “success” of a MOOC in terms of a large numbers of accesses (“open”), and hinder the “success of the students” (“course”) (in term of surveys and peer assignments actually submitted) who attended the MOOC. Another important piece of data regarding the “success of the students” for this MOOC is the number of answered quizzes. For the MOOC at the University of Melbourne, this number is 159874, which constitutes a percentage of 46% of the total number of students. Once again, we notice that “massive” informations are highlighted whilst the “course” ones are hidden. We know, in fact, that one student may submit more than one quiz, and we know that “a quiz” can be comprised of either of a single question, or a set of questions.

Figure 2: Miscellaneous from different MOOC infographics.
We can see that attention is paid to the features: “open” (ease of access), “massive” (integer numbers instead of percentages) and “brand”. The stress on these features and the absence of indicators that point towards “course” lead us to conclude that these two MOOC producers (i.e., Columbia and Melbourne) consider MOOC platforms as common social media, rather than carriers of knowledge. With regard to our starting point, we see possible resistance to change in the belief that “big numbers” are synonymous with “quality”.

In reports, guidelines and interview from FUN, the French MOOC producers claim to focus on the cognitive aspects of (Figure 3): pedagogical competences and pedagogical objects (“course”) are quoted in the first and second principles respectively. In the report they add that numbers shown on average on MOOC platforms are partial: “Note that much of the data related to a MOOC course is not available. The participants often rely on external resources or interact outside the dedicated space” (translated from French). For example, students watch a the video on YouTube, or interact outside the platform (“online”). Hence, the French MOOC producers argue that it is not easy to correctly understand the state of the art, and it is much more difficult to extract data concerning the cognitive processes. However, the French project focuses more on teaching (“course”) than the earlier mentioned MOOCs. In fact, the quality commitments are explained in a public document. In this document, the pedagogical aspects (“course”) regard the first and the second issues. On the other hand, in these guidelines a sort of empty toolbox is depicted: the specific contents do not matter, the toolbox can be applied to any subject. Didactical transposition is the responsibility of the individual lecturer. Hence, attention is paid only partially to didactical issues.

The French MOOC producers have wondered how to measure the efficacy, or the attendance of the MOOC course. In particular, “the first step is to understand the
goals and the target” of the course. They repeatedly underline the importance to test the course early (“course”), using quizzes and producing documents to support the course design. Matthieu Cisel, a PhD student at the École Normale Supérieure de Cachan which is dedicated to the development of the French MOOC, writes “the qualitative measurement doesn't mean to collect enrollment data, even if these ones measure the marketing efficacy”. He claims that it is necessary to maximize the active users percentage (“open”). We see a shift of attention from “ease of access” to “active users” in these words. Furthermore, we see that Cisel knows very well the population of students attending a MOOC, since he underlines also that active students change their status throughout their attendance to the course. He, thus, suggests to analyze the global dynamic: in particular, for each week it is possible to monitor the forum activity (“online”), the number of video viewed (both “massive” and “Video”), etc., of each student. He actually uses these indicators and on the official FUN (France Université Numérique) blog about quality, he underlines that certifications to students attending the MOOC can help measuring its efficacy: “I think that a xMOOC with ‘just’ 1000 participants and 30% of certifications is better than one with 5000 participants and 10% of certifications” (“scalability”). He ends the post on the blog by stressing that the multi-indicator (i.e., number of accesses + certifications) approach is very relevant. Moreover, in the report it is also explained that “the platform is the main point of access to the courses, so the visibility of the course and the platform’s visibility (“brand”) are strongly linked. […] the visibility of the courses may be also be useful for improving media visibility (“brand”)” (translated from French).

The private information about the Zurich MOOC reveal that Swiss MOOC producers interpret quality and efficacy in terms of “usefulness”: in fact, they have produced a MOOC to test a new didactic approach (“course”) named "flipped classroom”. The flipped classroom is a form of blended learning in which students learn new content online, and in class the teachers offer more personalized guidance and interaction with students, instead of lecturing. Thus, they have used their MOOC to change a traditional university course; in effect “rumors” say that the future of the Zurich MOOC project depends strongly on this experiment. The MOOC producers have measured the quality and efficacy by looking at final exam results (“course”). Clearly the Swiss project has paid considerable attention to didactical issues (“course”), and this case allows us to add a new indicator (with respect to our taxonomy): they call it “usefulness” and they measure it in terms of success on exams. Also this new indicator is a change with respect to the established standard. Outside the report, we read the use of the German language as an indicator of a “massive- target community”. Furthermore, “rumors” say that the Zurich producers were searching a way to reinforce their prestige keeping the university up to date (“brand”).

With regard to the Milan MOOC, we collected both private and public information. Similarly to the French project, the Italian project seeks for tools to measure efficacy, reachability and benefits. The goal of the Italian project is to bridge the gaps between
different educational levels (“course”), specifically to address the issue of transition from secondary school to university. The target is, as a consequence, the set of students who intend to enroll at university (“target community”). In this paper we report on findings regarding the issues brought to light by two distinct teams: the math-teachers team, and the multimedia team (“Video” and “Online”). Each team has its own view about quality: for the latter team “quality” is synonymous of “endearing videos” (“Video”), whilst for the former team it means “mathematical exactness”. Intertwining these different views, is the Italian definition of “quality”. The multimedia team has based the production of lectures on some research studies that discuss the quality of videos, the features that make them effective, and that maintain the viewer’s interest (Guo, Kim, Rubin, 2014). The multimedia team produced a document which summarizes each feature in detail: for example the optimal video length, the rate of words per minute, the format, the colors, etc. Furthermore, the multimedia team considers several platforms and discusses which one is the best to customize (“Brand”) and which one offers the best tools to allow students to interact (“online”) among themselves and with teachers. On the other hand, the math-teachers’ team has been interested in creating courseware to teach basic mathematical topics for the university; the team came to acknowledge that a change with respect to the teaching method was necessary (“course”). The team of lecturers make use of descriptors such as “understandable”, “easy”, “exact”, “linear”, “adequate”. They declare that they would use a not-refined but rigorous language (“open”). Quality for the Italian MOOC producers means: aesthetic, math exactness and usefulness.

Table 3 resumes the analysis described above, the columns contain the six variables, the rows contain the five MOOCs considered, each cell reports the keywords used by MOOC producers that identify the indicators related to the variables.

<table>
<thead>
<tr>
<th>Massive</th>
<th>Open</th>
<th>Online</th>
<th>Course</th>
<th>Video</th>
<th>Brand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Columbia [report]</td>
<td>Big numbers</td>
<td>Ease of access</td>
<td>/</td>
<td>Improvement educational outcomes</td>
<td>/</td>
</tr>
<tr>
<td>Melbourne [infographics]</td>
<td>Big numbers</td>
<td>Number of accesses</td>
<td>/</td>
<td>Number of surveys and peer assignment</td>
<td>/</td>
</tr>
<tr>
<td>FUN [report and interview]</td>
<td>Scalability</td>
<td>Active users</td>
<td>Interaction also outside platform</td>
<td>Pedagogical aspects</td>
<td>Frequency of video watched</td>
</tr>
<tr>
<td>Zurich [private]</td>
<td>Community target</td>
<td>/</td>
<td>/</td>
<td>Flipped classroom</td>
<td>/</td>
</tr>
<tr>
<td>Milan [private, guidelines]</td>
<td>Community target</td>
<td>Not-refined but rigorous language</td>
<td>Interaction</td>
<td>Teaching method</td>
<td>Specific team</td>
</tr>
</tbody>
</table>

Table 3: The indicators that describe the variables for each MOOC analysed. In square brackets in the first column indicates the type of information.
We can observe that there are some similarities and differences between attitude towards MOOC quality by each producer. Firstly, we notice that every MOOC producer takes into account “massive” and “course”. With regards to the term “massive” we point out that three approaches exist: the big number approach (Melbourne and Columbia), the community target approach (Zurich and Milan), and the scalability approach (FUN). With regard to the variable “course”, two teams approach the topic as the number of surveys submitted and improvement of educational outcomes (Melbourne and Columbia) focusing on numerical aspects, two other producers read “courses” as teaching method and the flipped classroom (Milan and Zurich), whilst the French producers focus on pedagogical aspects. Secondly, we point out that, using the above mentioned documents, some producers do not consider two variables: “video” and “online”. This is very interesting since MOOCs are based on a web platform and video lecturers. However, just two producers (FUN and Milan) take into account “online” using terms like interaction, whereas these two approach the quality of “video” differently: FUN uses frequency of video watched, whilst Milan relies on the skills of the multimedia team. Finally, we notice that some indicators are missing, such as “open” for Zurich and “brand” for “Melbourne”. This could be due to only having a partial data set.

Moving on to each producer, we can see that Melbourne approaches both “massive” and “course” using numbers as indicators. Columbia also uses a number approach in order to face “massive”, “course” and “brand”, in fact these three variables are linked by numerical aspects because the outcomes refer not only to number and visibility, but also to the number of people who attend and appreciate the courses. In addition, Columbia’s report stresses the importance of the ease of access (“open”) in order to ensure a lot of attendees. On the other hand, the producers of the Italian MOOC focus not only on a specific community and on new teaching methods, but also on how to customize the platform in order to allow the students to interact together. The “open” is understood as “not-refined but rigorous language”, thus pointing to the importance of creating a dialogue between teachers and students. FUN develops a peculiar approach to MOOCs. The French producers read “massive” in terms of scalability, and focus on pedagogical aspects. They also understand “open” as active users, highlighting the importance for students not to be passive in the interaction with the platform and hence showing that they have embraced interactionist approaches to learning (taking some distance from numerical aspects).

True, there is not a unique strategy in order to design and develop MOOCs, whereas it is clear that each producer designs its MOOC on his own, interpreting the variables in relation with the context (language, target, goals, etc...). As a synthesis, we could divide the five MOOCs into two groups: the first one (Columbia, Melbourne) minds numbers, the second group (FUN, Zurich, Milan) focuses on community and specificity, as well as on pedagogy and teaching. We can conclude that the first group considers a MOOC to be “good” if it is a “MOO” (rather hindering the “C”), while the second does not pay much attention on “M” and minds the “OOC".
CONCLUSIONS

We have focused on MOOC producers’ beliefs about quality. Literature on both MOOCs and beliefs about MOOCs is scarce, hence we have relied on general results in the field of affect and more specifically we have been inspired by Kontorovich (2013) to analyse the deeper meaning that MOOC producers assign to words such as “good”. To this end, we have collected a set of public and private information shared by MOOC producers and we used the words of the acronym “MOOC” to see whether they are taken into account, and which terms are specifically used to describe each variable. The result is a range of different sets of terms that describe different meanings assigned by MOOC producers to the notion of “quality”.

References


Documents and Reports


France Université Numérique (2013) - Engagements qualité des MOOC
DOI:

Melbourne University (2013). Melbourne University 2013 MOOC Infographic
This article presents a qualitative study on individual curricula on teaching analytic geometry and linear algebra on higher secondary schools focused on content and goals of educations. The individual curricula of nine teachers are analysed, compared, and classified into two types. The results are linked to didactical proposals, leading to the observation that the didactic community underestimates the potential of a traditional curriculum and that only one of the two types fulfils some didactic core proposals, like a focus on applications, a process-oriented view on mathematics, and an emphasis on “more general” goals of education, and not on specialised mathematical skills.

RESEARCH QUESTIONS AND BACKGROUND OF THE STUDY

This article presents a comprehensive overview of a qualitative study concerning secondary school teachers’ curricular beliefs on teaching analytic geometry and linear algebra on higher secondary school level.

The general framework of this research is seen in the global background theory of teachers’ beliefs following the idea that these beliefs are some of the major influence factors on the pupils’ learning (cf. Philipp, 2007) – an idea that has initiated a specific kind of research on beliefs in mathematics education (cf. Furinghetti & Morseli, 2010). More precisely, the study is based on in-depth interviews conducted by the author with nine higher secondary school teachers as interview partners. The interviews were evaluated according to the research programme of subjective theories, which is methodology invented by social psychologists to collect, to analyse, and to present complex systems of beliefs that are typically used by professionals as a background theory to structure and fulfil their job-related demands (cf. Groeben & Scheele, 2000.). The interviews were accompanied by (mainly two or three) classroom observations to verify if the subjective theories derived from the interviews are relevant to the teachers’ classroom practice.

INDIVIDUAL CURRICULA AND SUBJECTIVE THEORIES

Teachers’ professional demands highly consist of planning, performing, and evaluating their instructions against to the background of the relevant written curricula, using their pedagogical content knowledge as their individual horizons of
understanding both the written curriculum and its adaption to the specific situations found in their classes (cf. Shulman, 1986). Insofar, one of the teachers’ main tasks can be seen in the challenge of adapting a written curriculum to specific demands and circumstances of their classes according to their personal perception and pedagogical content knowledge. To describe the teachers’ adaptations of a written curriculum, the concept of individual curricula was invented to address a teacher’s beliefs system that forms an intermediate stage between the written curriculum and the enacted curriculum understood as the observable classroom interaction (cf. Eichler, 2007).

The concept of individual curricula follows the theory of curriculum transformation (cf. Stein, Remillard & Smith, 2007). It is based on the assumption that individual curricula consist of similar content as written curricula and that an individual curriculum is argumentatively organised in a similar manner as its official counterpart. Basically, that means that individual curricula contain beliefs about mathematical content, teaching methods, assumptions about the pupils knowledge and learning and the goals of education that should be achieved in the relevant teaching sequences. The argumentative structure of individual curricula is seen as a more or less coherent system of means-end relations subordinating the choices of mathematical content and teaching methods to the goals of education, prescribed by the written curriculum, but individually interpreted and adapted by teachers (cf. Eichler, 2007).

Following this brief outline on individual curricula, the research question of this study can be seen in the task of identifying the central beliefs on teaching analytic geometry and linear algebra and of structuring these beliefs in the sense of a curriculum. This task includes the demand to reconstruct the means-end relations between goals of education and the teachers’ opinions about choosing contents and methods. The outcome should not only consist of nine individual curricula, but also of a classification of these curricula and a comparison of them to commonly accepted standards within the didactics of analytic geometry and linear algebra.

**REMARKS ON THE METHODOLOGICAL BACKGROUND**

The programme of subjective theories, which was used to prepare and to analyse the teachers’ interviews, is based on the assumption that professionals use a more or less coherent system of beliefs to deal with their job-related tasks. This system is supposed to have a similar content, structure, and purpose as a scientific theory: A subjective theory contains general statements (similar to natural laws); the statements or propositions are logically ordered in hierarchies from basic assumption to general principals; and the content and its argumentative structure is used to derive explanation and forecasts and to solve technical problems (cf. Groeben & Scheele, 2000, para. 3). In case of teachers, the main “technical problem” consists of implementing adequate instructions to achieve the (individual) goals of education and forecasts and explanations are used to anticipate the effect on the pupils learning and to evaluate the teachers’ instructions of being successful or not.
A subjective theory is typically represented in a hierarchic diagram: On top, the most general goals and principals of the person who uses the theory are displayed; below these general goals, more specific ones are mentioned down to basic means to achieve the goals and principals above, using as many levels as appearing necessary to represent the structure of the subjective theory. The elements of the different levels are connected only to elements of a superior level. The connections are graphically represented by arrows and represent means-end relations that the teacher regards as valid between the concerned elements of the different levels (cf. Groeben & Scheele, 2000, para. 3).

The research programme of subjective theories includes a set of specific “dialogue-hermeneutic” methods to prepare the interviews and to evaluate the data. All these techniques consists of four steps (cf. Groeben & Scheele, 2000, para. 4): 1) The researcher composes the interview guide sequences. Each sequence starts with an open question, followed by structured questions that are derived from the relevant “objective theories”, in this case from didactic theories on teaching linear algebra and analytic geometry. The open question are used to make the probands speak freely and to address the topics they regard as important according to their own judgements. The structured questions are used to make the interviews more comprehensible and to address topics the researcher is interested in. 2) The interview is analysed by the researcher to identify and to isolate typical beliefs of the probands. This is the “hermeneutic” part of the methodology. 3) After the beliefs are identified, the researcher writes them down as short phrases on cards, and he meets the proband a second time to arrange these cards or phrases in a hierarchic means-end structure, using the probands as research partners and aiming for a consensus with them in representing their subjective theories graphically. This is the “dialogic” part of the methodology. This part of the methodology was invented to increase the validity of the interpretation and of the graphical reconstruction of the probands’ subjective theories (cf. Groeben & Scheele, 2000, para. 2 and 4). 4) Finally, the probands are observed to see if they behave according to the subjective theories reconstructed in the previous stages of the research process. To fulfil the last part of this process, the teachers are observed while teaching their lesson. The focus is not set on every detail on the classroom interaction, but merely on the general structure, e. g. on the choice of tasks, on the way of presenting mathematical content, and on the reactions on pupils’ success or failure.

After the subjective theories are reconstructed by the dialogic parts and validated by observations, they can be used for comparisons and classifications within themselves or to compare the content or the argumentative structure of the subjective theories to “objectives” theories that are related to the same topic.

In case of this study, the teachers were chosen randomly in 2013 by sending offers to twelve schools to invite one teacher of each school to take part in the study voluntarily. Nine of the schools responded, and the author moved to them to interview every single teacher in an in-depth interview of approx. 60 to 90 minutes.
length. About two months later, the author returned to set up the subjective theories’ structure with the particular interview partner. After discussing the representations of the teachers’ subjective theories, the author arranged two or three classroom observations to check the relevance of the subjective theory in practice. The nine teachers are referred to by the aliases “Alan” to “Ian”, using female names for women and male ones for men.

**DIDACTICAL THEORIES USED WITHIN THE STUDY**

In this study, didactical theories are used for two reasons: Firstly, they are the source of the structured questions used in the interview guide, and secondly, they are the background theories to compare the subjective theories not only to each other, but also to topics that are of interest within the didactical community. The didactical theories used within the study are basically connected to two topics: General goals of education in mathematics education and proposals for a “modern” or reformed kind of teaching analytic geometry and linear algebra.

In case of general education, a framework of Graumann is used. He distinguishes between five dimensions of general education (cf. Graumann, 1993, p. 195):

1) Pragmatic dimension: Mathematics education should be perceived as a useful tool to solve practical and technical problems of everyday life.

2) Enlightenment dimension: Mathematics education should foster an understanding of the world including its historical, cultural, and philosophical backgrounds based on mathematical theories and insights.

3) Social dimension: Mathematics education should strengthen the pupils’ competencies to cooperate, to communicate, and to accept responsibility.

4) Individual dimension: Mathematics education should enhance each pupil’s own abilities and interests.

5) Reflective dimension: Mathematics education should sensitise the pupils to the limits, boundaries, and fallacies of mathematical methods.

The didactical theory of analytic geometry and linear algebra is derived from Tietze’s comprehensive overview on contemporary approaches to this kind of mathematics education (cf. Tietze, 2000, especially pp. 149–158). Although being older than one decade and partly based on a previous work back in the 1980s, this is the last and still relevant major publication on this topic in the German speaking community. Tietze’s work is written in the spirit of Graumann’s understanding of “general education”, i.e. that the focus of teaching analytic geometry and linear algebra is supposed to be shifted from narrowly defined mathematical skills to more general insights and competencies in the sense of Graumann’s dimensions of general education (cf. Tietze, 2000, p. 151).

To be more concrete, Tietze demands the following changes in teaching linear algebra and analytic geometry: less algorithms and schematic tasks; more
exploration, problem solving, and the use of computer algebra systems to avoid too many and too difficult algorithmic tasks; less tasks of calculating intersections and distances of linear objects like points, straight lines and planes; more investigations of “geometrically rich” entities like spheres, conic sections, quadrics, and solids of revolutions; less axiomatic theory and pure mathematics; more explorative case studies and applied mathematics like stochastic matrices and population dynamics, subsumed under the concept of “modern applied linear algebra” (cf. Tietze, 2000, p. 155–158).

Tietze’s proposals are intended to address teachers directly, since the written curriculum of higher secondary schools gives room for individual choices, just a “core curriculum” of the intersection and distance tasks criticised by Tietze is set as obligatory.

Tietze’s main rationale for his proposals is based on the assumption that pupils should not become expert in axiomatic knowledge or calculating skills, but that they should obtain an insight in the role of linear algebra and analytic algebra in modern and scientific societies, that they should strengthen their general problem solving capabilities, and that they should understand the practical benefits of (computer based) linear methods in different scientific disciplines and even in everyday life (cf. Tietze, 2000, p. 151). These normative settings correspond to the five dimensions of general education in the sense of Graumann and can be regarded as a possible concretisation of Graumann’s ideas in the field of linear algebra and analytic geometry.

EXERPTS TAKEN FROM THE INTERVIEWS

This article does not provide enough space to describe the interpretation of the interviews and the reconstruction of the individual curricula in detail. Therefore, this chapter presents only some central excerpts of the interviews to make the main ideas of the data evaluation traceable.

EMPIRICAL FINDINGS I: THE ROLE OF THE CORE CURRICULUM

The most remarkable observation is related to the so-called “core curriculum” that Tietze criticised above (mainly tasks on intersections and distances of linear objects like points, straight lines and planes). All the teachers explain that these topics occupy a broad range of their curricula:

Brian: Well, basically metric geometry is such a comprehensive domain. Calculating angles, distances, relative positions, intersections, that’s all part of it, and additionally the various descriptions of geometric objects. And then, you need some algebraic tools like Gaussian elimination. And that is nearly the whole stuff. [...] Well, I regard it [analytic geometry] from a Cartesian perspective, well, and that’s a great advantage against Euclidean geometry. You are now able to make things calculable, and that is an important experience for pupils.
As Brian says, he does not only endorse the core curriculum traditionally or due to the written curriculum, but also to communicate the “Cartesian perspective”. Later, he mentions some additional reasons:

Brian: First of all, there’s a pragmatic reason – I have to admit –, [namely] the final secondary-school examinations. The less powerful pupils need to have some tasks to show that they can calculate, and in many technical occupations, algebra and calculating is important. [...] But if someone has learnt to deal with geometry problem in an analytic and algebraic manner, he has learnt a lot for his personality, spatial perception, and – I would place it into the focus – problem solving and argumentations. And if he [a pupil] has chosen mathematics as an advanced course, he should be prepared to study mathematics.

The first part of this excerpt would support Tietze’s criticism, since Brian stresses the algorithmic aspect of the core curriculum, but the second part introduces a different horizon: The tasks of the core curriculum are seen as suitable occasions to enhance argumentative and problem solving competencies. This view is supported by several other teachers, e.g. by Danna:

Danna: Basically, it’s the way to think. It’s the way to think that moves you forward and that is important beyond school. How shall I go on? That’s important, and not the [mathematical] result.

The teachers discussed some typical tasks they use for problem solving. Alan expressed what is most typical for the problem solving tasks in analytic geometry in his eyes:

Alan: Analytic geometry is the most beautiful part of mathematics. To get the insight that a product is zero, if something is vertical, that’s fascinating, and it’s more fascinating to see what you can do with this insight.

This statement seems to explicate what Brian addressed by the phrase “Cartesian perspective”, namely a correspondence and translation between geometric and algebraic concepts and descriptions. He regards this circumstance as the source of problem solving in analytic geometry

Alan: The distance from a point to a straight line in the three dimensional space is such a beautiful example. Two year ago, a pupil invented a method none of the teachers at our school had known before. That’s so brilliant, beautiful, and aesthetic. And if I can teach my pupils that there are six or seven methods you can find on your own – at least partly –, than that is beautiful to me and to the pupils as well.

To summarise the teachers’ curricular beliefs on the core curriculum: Besides some opinions Tietze criticised above, the beliefs are more many-sided as expected: There are in fact some influences of algorithmic tasks, but many aspects of general education Tietze intends to achieve by a revision of the existing curricula are pursued by the teachers within the context of the core curriculum as displayed in the following figure (fig. 1).
Fig 1: Core curriculum with goals of education

- **algebraic thinking in geometry**: promoting the Cartesian idea to describe geometric objects and relations by using algebraic expressions (A, B, D, F, I)
- **problem solving**: solving geometry problems by translating the problems into algebraic terms and by using the Cartesian idea to solve the problems by calculating (A, B, C, D, F, H, I)
- **coherence**: using a coherent set of concepts, skills, and methods within the whole syllabus (A, C, E, F, G, I)
- **algorithms**: posing a pool of algorithmic tasks that are easy to solve (A, B, C, G)
- **modelling**: using modelling task to illustrate the idealisation and limits used in analytic geometry to describe real-world situations (H)
- **fulfilling the written curriculum**: problems of intersection concerning linear objects; calculating distances of linear objects; calculating angles

**core curriculum**

- **individual and social dimension**: understanding mathematics as a product and theory, supporting interests to study mathematics
- **problem solving**: gaining general problem solving competencies and competencies to argue in intellectually challenging situations; possibilities to (re-)invent mathematics; enhancing spatial thinking and illustrating; possibilities to use social and "open" teaching methods like teamwork
- **coherence**: gaining insights into the role of algebra in geometry including its historical development and its use to describe and to solve geometry problems more easily or exactly than in Euclidean geometry
- **algorithms**: trying to get good marks even for less powerful pupils
- **enlightenment and reflection**: experiencing the difference between mathematical models and reality

**enlightenment**

- understanding the role of mathematics in science and society including its historical dimensions

**individual and social dimension**

- understanding the role of mathematics in science and society including its historical dimensions

**individual dimension**

- understanding mathematics as a product and theory, supporting interests to study mathematics

**individual and social dimension**

- understanding the role of mathematics in science and society including its historical dimensions

**enlightenment and reflection**

- understanding the role of mathematics in science and society including its historical dimensions
EMPIRICAL FINDINGS II: TWO TYPES OF INDIVIDUAL CURRICULA

The second finding of this study can be seen in categorising the teachers’ individual curricula into two types. As mentioned above, all the teachers use the core curriculum with different, but similar goals of education. Besides the core curriculum, the written curriculum opens room for individual extensions of the core curriculum. Therefore the whole individual curriculum can be seen as composed of the core curriculum and several extensions, each with its own goals of education (cf. fig. 2).

![Diagram representing the structure of individual curricula]

It is not possible to present all the nine individual curricula in detail. It seems to be more appropriate to summarise the observation that the teachers’ curricula can be classified into two types: The first type is more or less restricted to pure mathematics and includes only some physical applications of analytic geometry; the core curriculum takes a broad range, and there are few extensions; the goals of education are focussed on argumentative and problem solving competencies, on deeper insights in mathematical theories and methods, and on preparing the pupils to study mathematics or natural sciences. This “pure mathematics/physics type” is displayed in figure 3. Aspects of this type can be seen in the statements of Brian, quoted above.

The second type can be described as “applied-oriented and focussed on general education” (cf. fig. 4). This type does not want to expand the core curriculum, but is willing to extend it by different extensions; the extensions are not derived from a single topic, but from different fields where analytic geometry and linear algebra can be used; typically physics is avoided, and instead of physics, applications in social sciences, biology, and everyday life are used; many of these applications stresses the aspect of linear algebra, and not analytic geometry, whereas the first type only uses geometrical aspects. This kind of linear algebra is not understood in the sense of New Maths, but in the sense of “modern applied linear algebra, described by Tietze’ suggestions mentioned above.
The excerpts of the interviews quoted above represent the first type. To illustrate the differences to the second type, the limited space allows only one comprehensive quotation as an example:

Ian: I don't like to import physics into mathematics although physics is my second subject. The pupils don't like physics, and I think it's not our task to encourage the pupils to study physics or mathematics, which was a typical combination years ago. I think that's not our task. Well, but applications are important, though. But I don't want that they become experts. That's not the task of our general school system. They should get an insight in different applications of mathematics in different fields. I like matrices, population dynamics, not geometrical matrices. Next year, I will have to start with straight lines, planes, intersections and so on. But I am going to reduce this stuff to a minimum. It's too mathematical and a little bit boring, and I don't see the purpose of these topics.
CONCLUSIONS

The study presented here seems to have two main conclusions. The first one is related to the traditional core curriculum that is criticised by Tietze guided by the idea to establish more aspects of general education on higher secondary school level. The second conclusion concerns the ways of extending the core curriculum where two different types can be observed.

Firstly, all the teachers use a traditional core curriculum of calculating distances and solving intersection problems, and most of the teachers approve these topics. The core curriculum is often criticised from a didactical perspective for being too schematic, too algorithmic, and more or less incompatible with a “modern” curriculum focussed on general education and a broad range of not only mathematical competencies to be gained in mathematics education. In contrast to these objections, the place of the core curriculum in the individual curricula of the teachers interviewed in this study shows a very flexible use of the core curriculum and many links from its content and routine task to general goals of education. It appears as if the didactic community underestimates the potential of the traditional core curriculum.

Secondly, besides the core curriculum, the individual curricula of the teachers seem to be structured by extensions with different goals of education. It appears to be appropriate to distinguish between two different types among these extensions. The “pure mathematics/physics type” is very traditional and tends to extend the core curriculum in a coherent way, focussed on theoretical aspects of pure mathematics...
and including only some physical applications. This kind of curriculum seems to be related to a formerly common way of preparing pupils to study the “classical” combination of physics and mathematics at universities. The second type which is called “applied-oriented and focussed on general education” can partly be seen as an implementation of the proposals and reform ideas vivid in the didactic community. The core curriculum is reduced; several different, normally applied-oriented extensions introduce an overview of the many uses of analytic geometry and especially linear algebra in different contexts like in biology, sociology, and in everyday life aspects of mathematics. General education is predominant is the predominant aim. According to its various topics, the second type is less coherent as the first one in the sense of content and methods, and this incoherence is mentioned by the exponents of the first type as the most important disadvantage of integrating “new didactical ideas” into the established core curriculum.

Insofar, the teachers of the second type can be seen as proponents of reform ideas. But the didactical proposals typically include some additional aspects: case studies, extended use of computer based techniques, and the examination of “geometrically rich” object. None of these proposals could be found in the teachers' individual curricula, even not in the curricula of the second type of teachers. It would be interesting to clarify why only a part of the didactical proposals are accepted by the reform-oriented teachers. In this study, there was no rationale for this choice to be observable.

References


MATHEMATICAL REASONING AND BELIEFS IN NON-ROUTINE TASK SOLVING

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Linköping University, Linköping University and Dalarna University

This paper explores low performing upper secondary school students’ mathematical reasoning when solving non-routine tasks in pairs. Their solutions were analysed using a theoretical framework about mathematical reasoning and a model to study beliefs as arguments for choices. The results confirm previous research and three themes of beliefs are used by the student. These themes are safety, expectations, and motivation. The results also show a connection between beliefs and imitative reasoning as a way to solve non-routine tasks.

INTRODUCTION

The role of the affective domain in problem solving has been explored in several studies (e.g. Philippou & Christou, 1998; Schoenfeld, 1992; Callejo & Vila, 2009). It seems that beliefs can either be limiting such as the idea that certain tasks are connected to a specific algorithm (Lerch, 2004) or empowering such as showing persistence instead of surrendering when not knowing what to do (Carlson, 1999). Beliefs seem to interplay (Presmeg, 1993), and it looks like combinations of beliefs shape the way a student develops solutions to mathematical tasks (Furinghetti & Morselli, 2009). Sumpter (2013) identified three themes of beliefs when studying upper secondary school students’ reasoning and their arguments that were made when solving tasks: safety, expectations, and motivation. These themes of beliefs seem to interplay, for instance in students’ strategy choices when solving mathematical tasks. Aspects such as curriculum and the character of the task may influence the beliefs (Liu, 2010; McLeod, 1992). This implies that beliefs are highly contextualized and that studying beliefs demands a carefully chosen method of investigation. In Sumpter’s (2013) study the students were solving tasks similar to the ones in their textbook and the tasks tackled by the majority of Swedish upper secondary school students are seldom of non-routine character (Jäder, Lithner & Sidenvall, forthcoming; Sidenvall, Lithner & Jäder, in press). A majority of the students in the Swedish upper secondary school do not receive a higher grade than “pass” (Swedish National Agency for Education, 2013).

The research question posed is: What is the relationship between the beliefs indicated and the student’s reasoning when low performing students solve non-routine tasks?
BACKGROUND

This paper has two main concepts: beliefs and mathematical reasoning. In this section, we will present a short background of these concepts in the context of students task solving.

Beliefs

Seeing beliefs as being part of the cognitive domain and as an example of subjective knowledge (Callejo & Vila, 2009) helps us to distinguish this aspect from more short term developed aspects such as attitudes and emotions (Hannula, 2006). In this paper, we follow Sumpter (2013) and beliefs are defined as “an individual’s understandings that shape the ways that the individual conceptualizes and engages in mathematical behaviour generating and appearing as thoughts in mind” (ibid, p. 1118), e.g. one learns mathematics by solving tasks individually. We agree with Speer (2005) and see beliefs as attributions. By doing so we are aware of the impact methods and researchers play when we report data and draw our conclusions. Another issue to recognise is that it is not possible to establish causality between specific beliefs and their consequences (Callejo & Vila, 2009). Still the notion of beliefs works as a model that can produce attributions and thereby help with predicting and explaining behaviour. Following this, we use the notion of Beliefs Indications (BI) introduced by Sumpter (2013). BI is defined as a “theoretical concept and part of a model aiming to describe a specific phenomenon, i.e. the type of arguments given by students when solving school tasks in a lab setting” (ibid, p. 1116). This means that we talk about beliefs as being indicated and attributed.

When studying beliefs, one needs to separate between beliefs and motivation and emotions. We see motivation as active goals that could be intrinsic or extrinsic (Ryan & Deci, 2000), either positive or negative.

Reasoning

In order to study students’ reasoning we need a framework with a clear definition that focus on different types of reasoning which, but also allow the structuring of the various stages of the reasoning process so arguments for choices that are made can be highlighted. Therefore, this paper makes use of Lithner’s (2008) framework. Reasoning is then viewed as the line of thought adopted to produce assertions and reach conclusions in task solving. This definition provides flexibility when studying different types of reasoning since it does not have to be based on formal logic, and it even allows reasoning to be incorrect. Reasoning is a sequence, a product, which starts with a task and ends with a conclusion. We use the four step reasoning sequence proposed by Lithner (2008): (1) A task situation is met. (2) A strategy choice is made where ‘choice’ is seen in a wide sense (choose, recall, construct, discover, guess, etc.). (3) The strategy is implemented. (4) A conclusion is obtained.
The characterization of reasoning types is based on analyses of the explicit or implicit arguments for strategy choice and implementation. There are two main categories of reasoning: Imitative Reasoning (IR) and Creative Mathematically Founded Reasoning (CMR). In IR the task solver applies a recalled or externally provided solution method. In CMR the solver constructs a solution method. In the empirical studies behind the reasoning framework, Lithner (2008) identified three central aspects distinguishing CMR from IR: (1) A new reasoning sequence is created, or a forgotten one is re-created; (2) There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible; and, (3) The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning. IR contains no CMR, but CMR may contain parts of IR.

**METHOD**

Data was collected by recording video of task solving sessions and stimulated recall interviews, both of which were fully transcribed. The students’ written solutions were also collected.

Students were selected to represent programs with different intensity of mathematics. Adding to this, the selected students were expected to just pass the course. Teachers were asked to select students that usually work together in their task solving and are likely to communicate verbally with each other during the session. In order to stimulate verbal data, the students were asked to work in pairs, a common way of working in mathematics classrooms in Sweden (Swedish Schools Inspectorate, 2010).

To answer the research questions posed, we needed to identify and select tasks of a non-routine character that were in line with the course curriculum for the designated students. First we looked at the national tests in Sweden. They include tasks relevant for a specific course and therefore fulfilled the second requirement. The tasks from national tests were compared to the actual textbooks used, using a method of categorization previously applied in an empirical study by Boesen, Lithner and Palm (2010). The selection requirements were strengthened in that the national test tasks were to have no algorithmic guidance from the textbook. Four non-routine tasks at different levels of difficulty, all requiring CMR to be solved, were selected. The aim was therefore to provide a progression of difficulty as to meet each of the students at an appropriate level. We also added one routine task, solvable using IR, for the purpose of mirroring the results of the non-routine tasks.

The students participating in this study worked in pairs in a lab situation. They were placed in an adjacent room during an ordinary class session with a video camera and microphone set up. They had access to their mathematics textbook and a calculator.
In Swedish upper secondary schools the textbook is commonly used (Swedish Schools Inspectorate, 2010). They were encouraged to talk to each other while solving the tasks. Semi-structured interviews were conducted to clarify issues concerning the task solving sessions.

The procedure of analysis follows the same structure as Sumpter (2013). In applying the framework to analyse what reasoning a student used when solving a task, the first step was to structure the data using the four steps of reasoning to divided the students’ solution of the task into reasoning sequences of suitable grain size (Lithner, 2008). The reasoning sequence was then classified as comprising one of the reasoning types IR or CMR. To analyse students’ belief indications, a thematic analysis of the video-recordings, transcripts and students’ written solutions was conducted with equal attention to these data items (Braun & Clark, 2006). As a deductive approach, we used the three themes predefined by Sumpter (2013). Analysing the connection between BI’s and the reasoning used, we searched for patterns within the data.

RESULTS

Preliminary results indicate that the themes of beliefs indications identified when studying students’ work with routine tasks also apply on non-routine task solving. We will present a summary of the preliminary results (see Table 1) and then give an example of the analysis:

Table 1 The beliefs indicated and the reasoning used.

<table>
<thead>
<tr>
<th>Arguments</th>
<th>Themes of BI</th>
<th>Actions (including reasoning)</th>
<th>Number of occurrences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concerns are expressed about a lack of ability.</td>
<td>Intrinsic motivation</td>
<td>This is most likely leading to an urge for support by a peer-student which results in IR (instead of CMR which is abandoned).</td>
<td>3</td>
</tr>
<tr>
<td>Students express that a familiar algorithm is always the primary choice of approaching a task.</td>
<td>Expectations</td>
<td>The students’ ideas of how a task can and should be solved reflect an algorithmic approach (even to non-routine tasks).</td>
<td>6</td>
</tr>
<tr>
<td>Students express an urge for a solution that is not too complex, but also not too trivial.</td>
<td>Expectations</td>
<td>The students try to fit a solution of a certain level of complexity to the specific task depending on the context.</td>
<td>6</td>
</tr>
<tr>
<td>Insecurity is expressed in regard to the solution method.</td>
<td>Safety</td>
<td>This leads to the students drawing a conclusion to abandon (an even correct) CMR-solution and instead turn to a more a familiar strategy, IR.</td>
<td>4</td>
</tr>
<tr>
<td>When the whole solution is obvious to the student the strategy may include other aspects than mere algorithmic thinking.</td>
<td>Safety</td>
<td>CMR may be used is cases when the whole process is within reach for the student.</td>
<td>3</td>
</tr>
<tr>
<td>There is an urge to be able to present an answer within a limited amount of time.</td>
<td>Extrinsic motivation</td>
<td>To be able to present an answer students turn to the use of IR strategies.</td>
<td>5</td>
</tr>
</tbody>
</table>
Example of a task solving
A more detailed description of a student solving a non-routine task using the four phases of a reasoning sequence exemplifies how the method was used. The example presents how the categorization of types of reasoning was conducted, and the student’s arguments connect to belief indications.

Leila, who worked in pair with Anna, is trying to solve the following task:
“Which of the following expressions correspond to the perimeter of the figure?
\[ a + b \quad 2a + 2b \quad 3a + 2b \quad 3a + 3b \quad 4a + 2b \]
Justify your answer in your notebook.” (Swedish National Agency for Education, 2010, p. 3, authors’ translation)

![Figure 1](image)

**Fig. 1** Figure to task exemplifying BI’s and reasoning. (Swedish National Agency for Education, 2010, p. 3)

*Part 1.*

<table>
<thead>
<tr>
<th>Leila</th>
<th>[Reads the question]. Seriously, I don’t know how to do this stuff. […] What? Is this whole line (a)? [Pointing at the most left vertical line.]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anna</td>
<td>Yes.</td>
</tr>
<tr>
<td>Leila</td>
<td>And all this, is (b)? [Pointing along the bottom of the figure.]</td>
</tr>
<tr>
<td>Anna</td>
<td>Yes.</td>
</tr>
<tr>
<td>Leila</td>
<td>Isn’t2, or, eh, 2(a) plus 2(b)? Then it is… […]</td>
</tr>
<tr>
<td>Anna</td>
<td>Yes.</td>
</tr>
<tr>
<td>Leila</td>
<td>If you, [gesturing in the figure] just move these [indentations], kind of, then it will be, the you just take…</td>
</tr>
<tr>
<td>Anna</td>
<td>Yes..</td>
</tr>
<tr>
<td>Leila</td>
<td>No, I don’t get it. [pause] But it, if you, even if you squeeze it together, it should be 2(a) plus 2(b)? […]</td>
</tr>
<tr>
<td>Anna</td>
<td>Yeah. […] It should be that one [pointing at the expression 2(a) + 2(b)].</td>
</tr>
<tr>
<td>Leila</td>
<td>Mmh.</td>
</tr>
<tr>
<td></td>
<td>[…]</td>
</tr>
<tr>
<td>Leila</td>
<td>But then you could write, eeh: it is 2(a) + 2(b) since if you squeeze it together the sides becomes equal length.</td>
</tr>
<tr>
<td>Anna</td>
<td>Yeah.</td>
</tr>
</tbody>
</table>
Task situation 1: Which of the following expressions correspond to the perimeter of the figure and what is the justification?

Strategy choice 1: Leila argues that the indentations do not add any length to the perimeter of the figure. Therefore it can be considered to be a rectangle and the algorithm of a rectangle perimeter can therefore be used.

Strategy implementation 1: Leila adds the side lengths of the rectangle $a \times b$, (without considering that the indentations add to the perimeter).

Conclusion 1: Leila’s answers is $2a + 2b$, but hesitates in her justification of the chosen algebraic expression.

Leila’s reasoning: As a strategy choice, Leila uses the algorithm for computing the perimeter of a rectangle and by so doing she does not consider all of the necessary properties of the figure, the indentations. This algorithm is a familiar algorithm, and the reasoning is therefore categorized as IR. Leila’s conclusion from the first reasoning sequence results in a new reasoning sequence, since Leila hesitates when she is to justify her choice of expression.

Part 2

<table>
<thead>
<tr>
<th>Leila</th>
<th>Or? [pause] It can’t $4a$ plus $2b$, can it? [pause] Or, yeah, if you add these, [pointing at the vertical lines in the indentations] maybe. Look, this one...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anna</td>
<td>Uhmm.</td>
</tr>
<tr>
<td>Leila</td>
<td>... and then you add it to this one.</td>
</tr>
<tr>
<td>Anna</td>
<td>But what?</td>
</tr>
<tr>
<td>Leila</td>
<td>Look. [pointing at the figure]</td>
</tr>
<tr>
<td>Anna</td>
<td>Yes.</td>
</tr>
<tr>
<td>Leila</td>
<td>This, [pointing above of the left indentation] thing..., or, there is something missing here, ...</td>
</tr>
<tr>
<td>Anna</td>
<td>Yes.</td>
</tr>
<tr>
<td>Leila</td>
<td>... then you can kind of take this [pointing at the larger indentation] and this one [pointing at the smaller indentation] or, not. [Pause] I don’t get it.</td>
</tr>
</tbody>
</table>

Task situation 2: Which of the following expressions correspond to the perimeter of the figure and what is the justification?

Strategy choice 2: Leila considers the indentations as important for the computation of the perimeter. She also finds a way to integrate this necessary property into her solution. The argument supporting her following implementation is that the smaller vertical line of an indentation is the missing part of a vertical line of the larger indentation. The sum of these two vertical lines would be the same as the length of $a$. 
Strategy implementation 2: Leila adds one of the vertical sides of the larger indentation to one of the sides of the smaller one making up another $a$. She repeats this for the other side of the indentations which leads to Leila seeing four $a$’s in the figure. $a + a + a + a + 2b = 4a + 2b$

Conclusion 2: Leila’s answer is $4a + 2b$. Before Leila writes a justification for the choice of algebraic expression she states “I don’t get it.”

Leila’s reasoning: Leila creates, to her, a novel solution and she gives argument for her strategy choice (adding the vertical sides of the indentations) based on necessary intrinsic mathematical properties. Her reasoning sequence is therefore categorized as CMR. Leila abandons the correct solution resulting in a third reasoning sequence.

Part 3

<table>
<thead>
<tr>
<th>Leila</th>
<th>Should we write $2a + 2b$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anna</td>
<td>Mmh.</td>
</tr>
<tr>
<td>Leila</td>
<td>Because you, if you squeeze together the, block..., or, yeah, the figure,</td>
</tr>
<tr>
<td>Anna</td>
<td>It will be...</td>
</tr>
<tr>
<td>Leila</td>
<td>Mmh.</td>
</tr>
<tr>
<td></td>
<td>[Writes down the solution]</td>
</tr>
</tbody>
</table>

Task situation 3: Which of the following expressions correspond to the perimeter of the figure and what is the justification?

Strategy choice 3: Leila reconsiders her interpretation of the relevance of the indentations. She, once again omits the indentations by “squeezing” the figure. “Because you, if you squeeze the block[s]..., or yeah, the figure [...] then the figure becomes a rectangle.”

Strategy implementation 3: Adds the vertical sides ($a + a$) and the horizontal sides, interpreted as being of length equivalent to $b$ ($b + b$).

Conclusion 3: $2a + 2b$ with the justification: “because you squeeze the figure a rectangle is formed. Because a rectangle does not have as many sides as a square you can say that it is $2a + 2b$."

Leila’s reasoning: Leila returns to her incorrect answer $2a + 2b$ and omits an important intrinsic property of the figure; the indentations in her strategy choice. This reasoning sequence is therefore categorized as imitative reasoning.
Leila beliefs indicated

In the data, there are indications of several beliefs but here we focus on three main beliefs. The first one is an intrinsic motivational belief. When meeting the task, Leila expresses that she does not “know how to do this”. This reoccurs several times throughout the three reasoning sequences. Apart for being an indication of a negative intrinsic motivational belief, it could also be indications of a personal expectation: she does not expect herself solve the task. In the interview, when explaining her choices, she states on two occasions an insecurity regarding her own thinking: “I’m not sure if it is right, but [...]”. In the interview Leila also says that: “I am unsure when I have to think differently, therefore I choose the simplest way.” This statement indicates that there is an expectation on the task to be solvable in a way where the amount of thinking is reasonable according to Leila. The simplest way here, according to Leila is to regard the figure as a rectangle (without the indentations).

Leila’s reasoning and beliefs indicated

Leila uses IR in all four tasks. The first task is, according to Leila, a routine task [Post interview]. This is the only task where she arrives at a correct answer. Leila indicates similar intrinsic motivational beliefs expressing that she doesn’t understand how to do all four of the tasks. On two occasions she express an insecurity regarding her own ability. For instance, in the interview she states: “Because, I had probably done something wrong. You know, not knowing how to really think, and just skipped something [...] it had become more complicated”. In the example above, she has an expectation to not be able to solve the task. In the interview, Leila says that she “can only do tasks with normal shapes, and not with indents [...] it becomes too complicated. [...] You have to think differently, [...] but then I used what seemed easiest.” These three beliefs seem to interplay in a way that supports each other so that, for example the personal expectation of only being able to solve tasks with familiar geometrical figures may strengthen her intrinsic motivation of not understanding the task and also to her insecurity toward the task being of an unfamiliar character, and also toward her solution.

DISCUSSION

From the preliminary results, we can see that students, when working on non-routine tasks, use ding both CMR and IR. Compared to previous studies on student beliefs when solving routine tasks, this indicates that the task influences the way students reason (c.f. Liu, 2010). The results also indicate the same three themes of beliefs as in Sumpter (2013): expectations, safety and motivation, both intrinsic and extrinsic. Moreover, the beliefs seem to interplay and there is a connection between the students’ beliefs and how they solve non-routine tasks in this lab-setting. It appears
that students’ cognition, emotion and motivation are intertwined following the results from previous research (Furenghetti & Morselli, 2009; Presmeg, 1993). One of the main conclusions to be drawn from Table 1 is that students express beliefs that task solving does not include reflection or much struggle, CMR. There are indications of students expecting the solution to non-routine tasks to be algorithmic. This is in line with the findings of Lerch (2004). Data indicates that an algorithmic approach is the initial approach even to tasks considered by the students as non-routine, but the possibility to employ mathematically founded reasoning becomes limited by such a frame of mind.

Another expectation indicated is that students expect a task to be of a specific level of complexity. This expectation may connect to how the task solution is executed. In the example of task solving given in the results section Leila expresses “it felt too complicated”, when she was asked why she abandoned a correct solution method in favour of a, to her, less complex, but incorrect solution. In other situations students express that task solutions should be more complex than their initial attempt at a solution. The discrepancy between the expectations of a less or a more complex solution seems to be connected to the order in which the tasks are given. Leila’s task was the third task of five while on the fifth task (not presented here) several students indicated a belief that their solution was not complex enough.

Students’ expectations on a task solutions’ complexity are related to the idea that an imitative approach is more likely to be considered safe compared to non-routine tasks. In this study, we can see similar arguments in the conclusions; whether or not a CMR-solution is abandoned or not seems to be connected to the student’s level of security. If students express beliefs about feeling secure regarding the task solution, the solution is not abandoned, whereas if a student express an insecurity regarding the task solution, that solution is more likely to be abandoned for an IR-solution instead. This is in contrast to the successful students described by Carlson (1999) who persist working on a solution, not giving up, when not knowing how to proceed.

There are also belief indications of a will to always present a solution and an answer, even when students predict the answer to be incorrect and the solution to be unsatisfying. Leila’s peer Anna when working on the task in the example in the results section predicts her answer to be wrong. In the interview following the task solving session she expresses an urge to get the task done. She shows no considerations of a correct solution. A belief that an incorrect solution is better than a correct but not complete solution seems to lead to the students using IR.

It is probable to conclude that themes of beliefs interplaying with each other supports IR and makes it harder to change to creative reasoning that is anchored in mathematical properties. This means that when teaching problem solving, it is not enough just to change from routine tasks to non-routine tasks. We also need to teach what it means to construct mathematical founded reasoning.
References


Implications for teaching mathematics may be found by looking at the lives of those who have become successful at teaching and learning mathematics. This article presents results based on a qualitative narrative case study of three teachers’ self-reported past experiences with mathematics. In particular, the role of attitude and motivation in the teachers’ self-reported mathematical lives are considered. Support for the instability of attitude towards mathematics and for the manifestation of self-determination theory is found within the analyses of these cases.

INTRODUCTION

Teachers of mathematics hold a crucial role in the development of mathematical views among many young students. However, teachers were once students, and in their journey towards becoming mathematics teachers, they experienced various attitudes and motivations towards mathematics. The purpose of this study is to examine the role of attitude and motivation in the perceived mathematical experiences of teachers whose views and practices are in line with a contemporary understanding of what constitutes effective teaching.

In order to address issues of attitude and motivation in the described lives of teachers, theories and definitions developed by Di Martino and Zan (2010) around attitude and by Deci and Ryan (2000) around motivation are particularly pertinent. Although there have been many attempts at defining and redefining the construct of attitude in the field of mathematics education (Di Martino & Zan, 2001; Hannula, 2002; McLeod, 1992; Neale, 1969), the multi-dimensional definition developed by Di Martino and Zan (2010) is chosen as most appropriate due to its currency. Di Martino and Zan (2010) develop their definition of the attitude construct by compiling and analyzing data from more than 1600 students. They posit that attitude is determined through a dichotomy within each of the following components: emotional disposition, vision of mathematics, and perceived competence.

It should also be noted that attitude has been defined by McLeod (1992) to be relatively stable. However, Di Martino and Zan (2010) more recently claim that “it is never too late to change students’ attitude towards mathematics” (p. 27). A particular case of this may be found in Hannula’s (2002) ethnographic account of a lower secondary school student who’s attitude towards mathematics changes dramatically from negative to more positive within half a year. Various aspects of teacher effectiveness have also been found as having an impact on students’ attitudes (Haladyna, Shaughnessy, & Shaughnessy, 1983) and teachers’ influence on student
attitude has been addressed (Domino, 2009). It is therefore interesting to investigate whether or not teachers with a contemporary understanding of effective mathematics teaching perceive themselves as having changed their attitude towards mathematics throughout their mathematical lives.

Further, Deci and Ryan (2000) establish in their self-determination theory that satisfaction of the basic psychological needs for competence, autonomy, and relatedness is necessary to “facilitate natural growth processes including intrinsically motivated behaviour and integration of extrinsic motivations” (p. 227). This is another phenomenon that is important to identify within the reported lives of successful mathematics teachers because they have, in the end, become motivated towards teaching and learning the subject.

The intent of this paper is to present an overview of a small scale case study on the self-reported mathematical lives of three particular teachers. In what follows, methods and key theories are addressed, one particular case is presented, and results and implications are discussed.

METHOD

The three teachers participating in this study were randomly drawn from a 13 member cohort in the Masters of Secondary Mathematics Education Program at Simon Fraser University in Canada. Teachers in this cohort shared an interest in developing an understanding of the growing field of mathematics education and developing their own teaching methods in order to elicit deeper mathematical understanding in their students.

Each of the three participants (two females and one male coded as Tracey, Allyson, and Blake to maintain anonymity) were interviewed for approximately fifteen minutes according to a semi-structured interview approach that was loosely guided by the following interview script:

Although we are both teachers of mathematics, I’m sure we have both arrived at our current career quite differently. I’m interested in your story of your relationship with mathematics.

My parents took extra time to introduce mathematics to me at an early age. How was it for you?

By the end of high school, I told myself I’d never take mathematics again. Could you describe your experiences with mathematics throughout high school? Was there a point at which your relationship with mathematics changed?

At what point in your life did you decide you wanted to teach mathematics and why do you think this occurred?

Rapport was established with each participant, and the informal tone of the interviews was conducive to helping participants feel comfortable with being honest and open about sharing recollections of their feelings towards mathematics as experienced throughout their lives. It is not assumed here that these accounts
represent accurate depictions of participants’ past mathematical experiences, but rather, that participants are sharing experiences they recall as relevant to the discussion about their mathematical experiences.

Interviews were audio-recorded and transcribed. Analysis was guided by principles of narrative inquiry (Connelly & Clandinin, 1990). Narrative is “not simply a transparent recounting of events,” but rather “an organized interpretation of a sequence of events [with] temporal order” which includes “affective, relational, ethical and imaginative aspects of experience” (Bleakley, 2005, p. 536). It also “focuses on the way individuals present their accounts of themselves and views self-narrations both as constructions and claims of identity” (Burck, 2005, p. 252). Once narrative data is produced, it is a rich piece of qualitative data that can be remarkably informative and engaging for a reader.

The particular approach used in this study follows the principles of the life study method, where an analyst constructs an account from an interview transcription, retells the story to the participant, and then examines the narrative through qualitative recursive coding of themes. In the case of this study, the narratives are developed with the aim of constructing mathematical biographies (Kaasila, 2007). As Kaasila (2007) notes, the construction of a mathematical biography through narrative inquiry can be very informing of a participant’s self-perceptions, and often allows the participant to engage in retrospective explanation. Such explanations provide useful insight into the reasons for why a participant perceives that their mathematical identity, which includes attitude, has changed.

Once the narratives for each participant were created, participants were asked to read their narratives to check for accuracy and completeness. The process was iterated via email until the narratives were as complete and valid as possible. Recursive coding of narrative data produced themes and categories, which aligned with the theories developed by Di Martino and Zan (2010) regarding attitude and by Deci and Ryan (2000) regarding self-determination and motivation. These theories formed a conceptual framework according to which each narrative could further be analyzed and emergent themes could be derived.

**CONCEPTUAL FRAMEWORK**

Upon initial analysis of the narrative data, themes pertaining to attitude and motivation towards mathematics emerged. Statements that participants made regarding memories of past experiences with mathematics at various points in their lives clearly reflected a differentiation between various intensities of attitudes. There were statements that indicated profoundly negative attitudes, and statements that indicated profoundly positive attitudes. There were also statements made that indicated neutral attitudes.

In order to more clearly define positive, negative, and neutral attitudes, Di Martino and Zan’s (2010) multi-dimensional definition of attitude is used to inform coding of attitude related responses by participants according to positive, negative, and neutral.
Each of Di Martino and Zan’s (2010) components (emotional disposition, vision of mathematics, and perceived competence) can be viewed as either positive or negative. In particular, emotional disposition can be coded as positive when language used points to contentment, and negative when language points to disturbed emotions. Vision of mathematics can be coded as positive when language refers to mathematics viewed in a relational manner, and negative when language refers to mathematics viewed in an instrumental manner. Finally, perceived competence can be coded as positive when the participant claims they were successful, and negative when they express a sense of failure. In the analysis, statements made by participants pertaining to attitude were coded as positive if all three of the abovementioned dimensions were classified as positive, negative if all three of the abovementioned dimensions were classified as negative, and neutral if only one or two of the dimensions were classified as positive.

In terms of motivation, it was evident in the initial analysis that there were nuances between the various sources of motivation that participants were referring to in their recollections of experiences with mathematics. However, it was difficult to classify these nuances. Therefore, Deci and Ryan’s (2000) motivation-related constructs were selected for the analysis. This means that each of the three basic psychological needs of competence, autonomy, and relatedness were coded as either being satisfied or unsatisfied, and elements from the taxonomy of human motivation (as outlined in Table 1 below) were identified in the narrative data.

![Table 1: Taxonomy of human motivation (Deci & Ryan, 2000, p. 61).](image)

A summary of all the factors used in the analysis of participant narratives is provided in Table 2 below.

![Table 2: Classification of factors used in analysis of narratives.](image)
In addition to these factors, the theoretical framework developed by Di Martino and Zan (2010) for a multidimensional model of attitude, where “emotional disposition towards mathematics, vision of mathematics, [and] perceived competence in mathematics” (p. 44) are the key components of attitude towards mathematics, is also considered in the analysis.

**RESULTS**

From the analysis, five key themes emerge: instability of attitude, effects of perceived competence on attitude, effects of vision of mathematics on attitude, effects of low levels of autonomy on motivation, and support for self-determination theory. For purposes of brevity, one particular case, chosen for its strength to represent all five emergent themes, is presented in what follows.

**Case of Tracey**

Tracey was introduced to mathematical concepts informally at a very young age by her academic parents. Her brilliant father would talk about the mathematics used in building a bridge with fascination as he drove over a bridge. Tracey and her sister were also taught about the value of money and how to make change at a relatively young age. Her father would give mathematical problems at the dinner table and would never give out the answer. Tracey would struggle for days to figure these problems out while her sister chose not to engage in them. Mathematics was always discussed in daily family life even though it was not necessarily introduced as mathematics.

Since Tracey’s parents were academics, they valued success in education. It was very important to her father that Tracey and her sister did well in school. However, it was their responsibility to ask for help when they needed it. Tracey found it very frustrating to ask her dad for help because she didn’t feel like he could help her. He was smart, but was unable to articulate how he got the answer a lot of the time. Her parents held high expectations for Tracey and her sister’s success in school, and Tracey fulfilled that expectation. They were disappointed in her when she did poorly and she didn’t want them to be disappointed. She would go through all of her corrections every time and was for the most part always successful in school mathematics. Even though she was stressed going into tests, she considers herself lucky to have done well the majority of the time. She was proud to bring home high scores.

Tracey liked math because she knew what to do to get the right answers and enjoyed bringing her work home to show her parents. Although other subjects were important, math was most important for her father because he was passionate about it himself. Interestingly, Tracey’s sister did not share Tracey’s positive attitude towards math. Tracey was the one who took on her father’s passion for mathematics.

Tracey succeeded in and enjoyed school mathematics all the way through grade 10. In grade 9/10, she had a teacher who she thought of as really smart. This teacher
taught in a traditional manner, and she liked that. At the end of grade 10, she applied for and was accepted into the baccalaureate program for grades 11 and 12. This was when she no longer stayed at the top of her class in achievement because she was now being compared to a whole class of really good students. This experience defeated her confidence in her own mathematical abilities. She felt like she had thought all along that she was good at math, but now that she wasn’t the best student in her class anymore, she convinced herself that she really wasn’t that good at math. Her parents were also disappointed in her for not being the best in her class. Further, the teachers in these grades didn’t take the time to explain things very well. They left students to figure things out on their own.

Tracey felt like she had no guidance and this made it even more frustrating of an experience for her. Even though there was time in the classes for study groups, her confidence was so low that she opted to pretend she understood in the groups so that she wouldn’t look foolish. She considered going back into a regular non-baccalaureate math course in grade 12, but she didn’t want to quit. That’s just not something members of her family did. She persevered through the baccalaureate program through grade 12 by memorizing required processes to get through. Her vision of math became formed by this experience. She worried that her marks wouldn’t be high enough for university, and when she finished grade 12, she had lost her confidence in her mathematical abilities. She did well, but was not top in her class.

Feeling defeated by her experiences in the baccalaureate program, Tracey opted to become an elementary school teacher rather than a high school teacher because she just couldn’t see herself teaching high school math. However, as time went on and she began teaching elementary school, she noticed that she came alive whenever she taught math. It was exciting for her, and she found herself passionate about it in front of her class. She began tutoring high school level math. She had the drive to show these students just how fun and interesting math can be. Teaching reaffirmed for her that she could be a good math teacher and that she could teach the high school curriculum because she did indeed understand it. She remembers that her dad, although brilliant at math, was not a very good teacher. She can see herself surpassing his teaching abilities, and now wants to retrain to teach high school math. Her goal is to change high school kids’ views of math and wants them to leave her class thinking that math isn’t all that bad.

**Analysis of Tracey’s Case**

Based on the narrative data constructed from Tracey’s reports on her mathematical life, Tracey maintained an overall positive attitude towards mathematics until grade 11 and 12 when she was placed in a more advanced level and she was no longer achieving as highly as compared to her classmates. Her motivation was intrinsic before school, but soon became introjected intrinsically due to her focus on comparative achievement. Her achievements caused her to enjoy doing mathematics.
Late secondary was a pivotal point for Tracey because this is where her attitude towards math plummeted due to low perceived competence. It took her a while to recover from this event.

In what follows, Tracey’s perceived relationship with mathematics is analyzed according to each factor described in the conceptual framework within the following periods of her life: preschool, elementary, early secondary, late secondary, post-secondary, and career.

Before school, Tracey was intrinsically motivated to complete mathematical tasks. This is evidenced by her joy of solving problems given to her by her father and her interest in her father’s mathematical observations in daily life. All three of her basic psychological needs were satisfied during this time thanks to her family’s positive outlook on mathematics. Tracey adopted her father’s positive attitude towards mathematics.

All through elementary and early secondary school, Tracey focused on achievement. This was influenced by her parents who held high expectations of her achievement. In the interview, when talking about this time period, Tracey said, “I liked knowing what to do, getting the right answer and bringing it home.” She developed the viewpoint that mathematics was procedural, and she focused on learning the steps in order to succeed. In this regard, her motivations were introjected extrinsically because of her focus on approval by her family. It is interesting that Tracey’s attitude remained positive throughout this time even though her vision of mathematics was incomplete. This was in part due to the success that she had with mathematics. Her needs for competence and relatedness were being satisfied because she was receiving high grades and her family was happy with her successes. However, she did not have much autonomy over the mathematical tasks she engaged in and therefore was dependant on a somewhat external source of motivation.

With all the success she had in mathematics until grade 10, Tracey enrolled in a baccalaureate program in late secondary school and it caused a big shift in her attitude based on her recollections. Her basic needs of competence and relatedness were not being met and it was the satisfaction of these very needs that had motivated her towards excelling in mathematics up until then. She still had autonomy because she chose to enroll in the more advanced level, but it was not serving her in the way that it should have. The loss of perceived competence and relatedness caused her attitude towards mathematics to become more negative. Further, she began to see mathematics as a set of procedures because she was focused on getting through the material rather than fully understanding it. She was motivated by her desire to get into university and she knew she needed high enough grades to attain this goal. This is evidence of identified extrinsic motivation because she claims that she consciously valued completing math work at this time with the ultimate goal of entering university. Her attitude shift, and more specifically, her lowered perceived competence made an impact on her career choices.
In post-secondary, Tracey chose to avoid high level mathematics courses because she didn’t think she was very good at math. Her needs of competence in mathematics were not being satisfied because she didn’t enroll in difficult mathematics courses. In fact, she reports that she chose to train to become an elementary teacher rather than a secondary teacher because of her low perceived competence levels. Her attitude was still recovering from the shift she encountered in late secondary school. However, her needs for autonomy and relatedness were being satisfied because she had choice over what she wanted to do and her family was generally supportive. Although she didn’t pursue becoming a secondary school mathematics teacher, she pursued knowledge of mathematics for the purpose of attaining her goal of becoming an elementary school teacher. This contributed to her integrated extrinsic motivation towards learning mathematics.

Finally, when she was teaching mathematics in elementary school, she reports that teaching mathematics was the most enjoyable part of her job. She also started tutoring secondary level mathematics, which enforced for her the fact that she was indeed competent in mathematics. With her perceived mathematical competence flourishing again, she became intrinsically motivated to study further and to retrain to become a secondary mathematics teacher. This was an autonomous choice and she had the support of her friends and family. With all three psychological needs being met, her expressed attitude towards mathematics became positive again and her motivation now became internalized into intrinsic motivation towards learning mathematics.

To summarize Tracey’s expressed relationship with mathematics, a visual representation of the factors described above in presented in Figure 1 below. This is not the data itself, but rather a representation that enables comparison of factors both within each participant’s narrative and across the participants’ narratives in a visual format. Values used to generate the visual representation reflect the coding of each factor within each time period referred to in the narrative data. That is, the three psychological needs were assigned a value of 1 for unsatisfied and 6 for satisfied. Attitude levels were represented by a value of 1 for negative, 3 for neutral, and 6 for positive. Likewise, motivation levels were represented by a value of 1 through 6 for each of the six regulatory styles in the taxonomy of human motivation in the order presented in Table 1. These regulatory styles are also noted below each time segment in Figure 1 for added clarity.
Figure 1: Tracey’s Expressed Psychological Needs, Attitudes, and Motivations in Mathematics.

**Discussion**

Tracey’s case provides an example of the five emergent themes of this study: instability of attitude, effects of perceived competence on attitude, effects of vision of mathematics on attitude, effects of low levels of autonomy on motivation, and support for self-determination theory.

Although Tracey’s expressed attitude towards mathematics was interpreted as negative in her experiences with mathematics in late secondary school, it shifted back to being positive when she moved into her career as a mathematics teacher. Such shifts in attitude were also evident in other narratives in this study, and together affirm that attitude towards mathematics can change.

It was also found that participants in this study evidenced links between perceived competence and expressed attitude. Tracey’s case is representative of this finding because her expressed attitude became negative during a time when her perceived competence was being compromised. Interestingly, even when the basic need for autonomy was not being met and she was motivated more extrinsically, perceived competence seemed to play a role in her attitude towards mathematics. This implies the strength of perceived competence and its potential effects on attitude towards mathematics.

Links between vision of mathematics and attitude were also evident. All participants, including Tracey, experienced a time during which they viewed mathematics as arbitrary, “made up,” and primarily a subject where procedures needed to be memorized. While discussing these times, participants exhibited neutral or negative attitude towards mathematics. For two of the participants, this occurred when referring to their experiences of mathematics classes in late secondary school.
Another interesting theme that emerged was the relationship between low autonomy and motivation. Every time a participant identified with needs of autonomy being unsatisfied, even if competence and relatedness were satisfied, they were prone to being more extrinsically motivated than intrinsically.

Finally, data collected in this study supports Deci and Ryan’s (2000) claim that satisfaction of the basic psychological needs for competence, autonomy, and relatedness is necessary to “facilitate natural growth processes including intrinsically motivated behaviour and integration of extrinsic motivations” (p. 227). More extrinsically motivated behaviour occurred when one of these needs were unsatisfied.

CONCLUSION

In sum, an analysis of narratives constructed from three mathematics teachers’ self-reported mathematical lives illustrates that the three basic needs of competence, autonomy, and relatedness, when satisfied simultaneously, promote a more internalized level of motivation towards learning mathematics. And also, that a positive vision of mathematics along with the satisfaction of these needs promotes a more positive attitude towards mathematics. These findings give support to the notion that attitudes towards mathematics can change, and that self-determination theory plays a supportive role. Although data for this study was taken from a small population, the findings from the analysis of the in-depth narratives produced from this population have implications for teaching practices across the field of mathematics education. Most importantly, there should be a greater emphasis placed on satisfying student needs for autonomy.

References


STUDENTS REASONING AND UTILIZATION OF FEEDBACK FROM SOFTWARE

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This paper is an extract from a study investigating the way students’ reasoning and utilization of feedback relate to success and failure in task solving. Sixteen 16-year old students solved a linear function task. They were instructed to use GeoGebra as a mediator and they had the responsibility to choose solution strategies. The results were analyzed using Lithner’s (2008) framework of imitative and creative reasoning together with Shute’s (2008) definitions of formative feedback. Schoenfeld’s (1985) protocol analysis was used to structure the path through solving the task. The results showed that students who were successful in solving the task reasoned creatively and used feedback elaborately. This paper focuses on frameworks and the method of the study.

INTRODUCTION

Two students tried to create a vertical line in the graphic-field of a dynamic software, GeoGebra, by submitting algebraic expressions (\(y=mx+c\)). Their strategy had been to increase the \(x\)-coefficient until the slope became vertical. “We need a really large one, let’s put in \(y=100x+1\)”…. they performed the activity, interpreted the feedback from GeoGebra, and found that their strategy didn’t work… “Wait, if they are vertical they should not intersect the \(y\)-axis at all….”. Apparently they predicted incorrectly the result of the activity, they received feedback from the software, and they elaborated on the result of the activity, and drew a correct conclusion. It seems like dynamic software such as GeoGebra offers a guide to the students’ task solving in the sense that they are invited to set up target images for their actions, and the computer’s precise feedback of the action offers possibilities to interpret and elaborate on ideas for subsequent actions. Research has shown, that student discussions often are mathematically shallow when they are solving tasks. One reason may be that in regular teaching, students are not encouraged to create original solution-methods; instead they are guided into rote-learning strategies as they are provided with examples and formulas by instructions (Hiebert & Grouws, 2007; Lithner, 2008).

Considering rote-learning, it’s important to investigate the causes, consequences, and alternatives. In the perspective of reasoning Lithner (2008) defines imitative reasoning (IR), which is related to rote thinking and its opposite, creative mathematical reasoning (CMR), which is characterized as creating original methods of solving a problem which are supported by argumentation anchored in mathematics. A study (Jonsson,
Olsson, Liljeqvist, Norqvist, & Lithner, 2014) showed students practicing CMR learned better than those who practiced IR. On the assumption that CMR is better for learning Granberg & Olsson (2015) performed a study investigating the way interactive software (GeoGebra) supported CMR. It was found that GeoGebra guided students into creating goals, planning activities, receiving feedback, and evaluating the result of the activity. In the present study, the way of using feedback and the associated evaluation is further investigated through questions about the relationship between the students’ reasoning and their use of feedback generated by GeoGebra. Therefore the aim of this study is to investigate the relationship between the students’ reasoning and the way students utilize the feedback provided by GeoGebra.

**AIM AND RESEARCH-QUESTIONS**

The aim of this study is to develop understanding about students’ utilization of feedback from software, associated to their reasoning and success in task solving during joint problem solving aided by GeoGebra.

The research questions guiding this study are:

- What is the relationship between the students’ way of using the feedback that GeoGebra generates and the students’ reasoning?
- How do the students’ ways of reasoning and utilization of feedback from GeoGebra relate to their success in problem solving?

To examine the students’ reasoning and the utilization of feedback generated by GeoGebra, a didactical design (which will be presented in detail later) used in a previous study (Granberg & Olsson, submitted) was adopted. It was designed in line with didactical propositions of Brousseau (1997) and Schoenfeld (1985). Student dialogues, gestures, and screen activates were recorded and used as data.

**FRAMEWORK AND BACKGROUND**

The main components of the research questions are reasoning, feedback, and success of problem solving. The research questions concern the relationships between those components. To structure data Schoenfeld’s (1985) framework for protocol analysis was used. To answer the research questions, concepts of Lithner’s (2008) framework was used to analyze reasoning and concepts of Shute (2008) were used to analyze feedback. The paper ends in a discussion of the results from the perspective of the students’ attitudes to mathematics referring to Di Martino and Zan (2010). Each part of the framework and background will be further presented in the following paragraphs.

**Problem solving**

Schoenfeld (1985) elaborated and extended Pólya’s (1945) four problem-solving phases to the following six: Reading the task, analysis (why the properties of a task have certain consequences), exploration (why some outcomes will be useful), planning (why a certain approach would lead to solution), implementation (why the search for a
solution proceeds in a proper way) and verification (why a solution is actually reached). Focusing on decision-making at the executive or control level, Schoenfeld (1985) proposed a method of protocol analysis to examine the way decisions shape the path through problem solving. The protocol is based on the six phases of problem solving and the transitions between these phases. Protocols are parsed into episodes, which are periods of time during which the problem solvers are engaged in a single set of action of the same type or character such as planning, exploration, implementation, etc. Three classes of potential decision points are described; the junction between episodes, when new information or possibilities to take a new approach occur to the student, and when difficulties indicate that consideration of a change in approach is needed. In the present study Schoenfeld’s framework will be used in order to structure the students’ attempted solutions into episodes, and to consider whether certain decisions may be related to successfully finding a solution. The conversations and computer activities associated with determining a successful solution will be further analyzed through Lithner’s (2008) framework of reasoning.

Reasoning

Students who are attempting to solve mathematical tasks will engage in reasoning. Lithner (2008) defines the learner’s reasoning as her line of thought, that is, the thinking process during which the learner successfully or unsuccessfully attempts to solve a mathematical task. Reasoning is guided, as well as limited, by the student’s competences and is created in a sociocultural milieu. Lithner characterizes reasoning as imitative or creative.

During their quest for a solution to a given task, the students’ strategy may be to recall known facts, algorithms, or procedures that can be followed in order to reach an answer. Lithner (2008) associates these strategies to imitative reasoning, IR. One variant of IR is memorized reasoning, to recall memorized facts or complete answers, e.g. a proof, a definition, or that 1 liter = 1000 cm³, but most school mathematics tasks ask for some kind of calculation or other process and such tasks can often be solved by algorithmic reasoning, to apply provided or memorized procedures and algorithms. This is often efficient if the algorithm is remembered correctly and then only a careless mistake may prevent the student from reaching a correct answer. Imitative strategies are described as memorizing and recalling, and often lead to rote learning.

Creative mathematical founded reasoning (CMR), is characterized by novelty, plausible argumentation and mathematical foundation. That is, instead of recalling a procedure that will solve the task, the students’ create methods that, at least to some extent, are new to them. The solution strategies may be supported by plausible argumentation, which is anchored in intrinsic mathematical properties of the involved mathematical components. Lithner (2008) suggests a wide conception of mathematical reasoning. In contrast to strict mathematical reasoning, which means distinguishing a guess from a proof, plausible reasoning includes also distinguishing a guess from a more reasonable guess. Plausible reasoning is not necessarily strictly logical but
constructive through the support of plausible arguments. The more plausible they are, the stronger the logical value.

In order to address the question of what an argument is, Lithner (2008) introduced the notion “anchoring”, which refers to its fastening in relevant mathematical properties of the components about which one is reasoning: objects, transformations, and concepts. The object is the mathematical component, the transformation is what one is doing with the object (a sequence of transformations is a procedure), and the outcome is another object. A concept is a mathematical idea that builds on objects, their transformations, and their properties. Depending on the purpose of a transformation, a mathematical property may be superficial or intrinsic. Lithner (2008) illustrates this in the following example (p.261): *In deciding if 9/15 or 2/3 is largest, the size of the numbers (9, 15, 2, 3) are a surface property that is insufficient to consider while the quotient captures the intrinsic property.* If the student, instead of applying a memorized procedure creates an original solution method (provided it’s not done by pure guesswork) it is necessary to construct arguments for why the method will solve the task. Argumentation may be considered as predictive or verificative. Relating to Schoenfeld’s problem solving phases presented above, in the phases of analysis, exploration, and planning, the arguments are primarily predictive. The phases’ implementation and verification include primarily verificative argumentation (Lithner, 2008).

**Feedback**

The students’ activities in GeoGebra may have a more or less articulated purpose of determining something particular. The actual computer activity, when the student’s input is entered and the result of GeoGebra’s processing appears on the screen, generates feedback associated to the action. In this study it is assumed that the computer activity is intended to contribute to the solving of the task and the information from GeoGebra is feedback. It is also assumed the student will use the feedback in different ways, e.g. to determine if they are right or wrong, to find clues how to proceed, etc.

According to Shute (2008), information meant as feedback to a learner in response to some action on the learner’s part can be delivered in different ways, e.g. verification of response accuracy, explanation of a correct answer, hints, worked examples, and can be administered at various occasions during or after the learning process. Feedback directed to the student’s activity is considered as having an effect on student learning. This is known as *Formative Feedback* and has the purpose of promoting learning (Shute, 2008). Shute’s definition of formative feedback is *information communicated to the learner that is intended to modify her thinking or behavior for the purpose of improving learning* (p.154). In this study the feedback from GeoGebra is a result of an activity planned by the students. The student may have an idea of what feedback she needs, she will shape the computer activity in relation to that purpose and may have the opportunity to use feedback from the software to modify and improve her learning.
Shute (2008) found that a specific form of formative feedback, *Feedback on task-level*, is particularly effective for supporting learning. Compared to general summary feedback, feedback at the task level is more specific and often provides real-time information about a particular response to a problem or task to the student. In this study the feedback is considered as at the task-level.

Formative feedback consists of two parts affecting each other. In learning situations a teacher may give a response that is dependent on a student’s behavior, which in turn may affect the student’s behavior. Brousseau argues that feedback does not necessarily come from a teacher or a peer; it may be a result of the student acting on the learning situation, which in turn will change as a result of the action. If the learning situation changes the student has to reconsider her behavior (Brousseau, 1997). In the current study, one of the main parts of the learning situation is the interface of GeoGebra. The dynamic software will respond according to the student’s activity and in turn affect the students’ actions. This will be considered as an example using feedback from the interactive software.

Formative Feedback provides students with two types of information: *Verification* and *Elaboration*. Verification involves confirming whether an answer is correct or incorrect and can be accomplished in different ways; explicit, e.g. a prepared piece of information from a teacher or, implicit, e.g. expected or unexpected results in a simulation. Elaboration has several variations, e.g. to address the response, discuss particular errors, provide worked examples or give gentle guidance. One type of elaboration, response specific feedback is considered as particularly learning-efficient. Response specific feedback focuses on the learners answer and may describe why or why not an answer is correct (Shute, 2008). In this study the feedback from software is considered as implicit and both verificative and elaborative. If the students have articulated a prediction of the outcome of an activity and just note whether the prediction is fulfilled or not, it is defined as verification. If the students discuss the outcome in terms why the results corresponded, or failed to correspond, with the predicted results or if the outcome is elaborated in some other way (above just noting if a prediction has been fulfilled) it is considered as elaboration on the feedback.

**Students’ attitudes**

Di Martino and Zan (2010) suggest instead of just explaining mathematics difficulties with negative attitudes, teachers should identify a student’s attitude profile in particular negative attitudes associated to mathematic difficulties. Instead of stating the difficulties depend on attitudes, teachers could act towards a change. Di Martino and Zan (2010) investigated a large number of student narratives about mathematics and found that attitudes often change over time and that this change is due to various situations.
METHOD

The method was designed to answer the research questions about reasoning, feedback, success of solving the task, and the relationship between those components. To collect data, a didactic situation, previously used by Granberg & Olsson (submitted), was adopted. In that study the didactic situation was found to engage students in reasoning and to use feedback generated by GeoGebra.

The didactic situation

The didactic situation was built on three propositions: challenge, responsibility, and collaboration. Schoenfeld (1985) argues that students need to work with mathematical problems that to some extent are new to them in order to develop problem solving skills. Brousseau (1997) proposed that if a task shall remain a challenge the students must have the responsibility to create their own methods for arriving at a solution. Furthermore Brousseau suggests that the teacher should instruct students until they can continue on their own, and then devolving the responsibility for solving the task to the students. During student-active sessions the teacher should not guide the students to right answers. If a task has an appropriate design, the students will reach the target knowledge for the task if they solve it. If the teacher offers information regarding how to solve the task, the students will not reach the knowledge target.

Working in small groups has been reported beneficial for learning under the circumstances that the task is focused on relationships and concepts rather than procedures. The former invites to collaboration and the latter to co-operation (Lou, Abrami, & d’Apollonia, 2001; Mullins, Rummel, & Spada, 2011). Collaboration is understood as a coordinated activity that is the result of a continued attempt to construct and maintain a shared conception of a problem (J. Roschelle & Teasley, 1994). In contrast, co-operation means that the co-operators split the task into parts and each one works with different parts.

The students worked in pairs sharing one computer using the software GeoGebra. The task consisted of creating three pairs of linear functions whose graphical representations where perpendicular and to formulate a rule for the circumstances when the graphs of two linear functions are perpendicular.

Sample and procedure

Sixteen students from the science program at a Swedish upper secondary school volunteered. They were 16 – 17 years old, 8 girls and 8 boys. They had some earlier experiences with linear functions but they had no recent instruction regarding the issue.

The students solved the task outside the classroom in pairs. They used a prepared GeoGebra-file, which contained a textbox with the instructions for the task and all tools were disabled except for the pointer, the “layer-mover”, and the angle-tool. They had a short introduction to GeoGebra, how to submit formulas into the input-field, how to change an algebraic expression and how to use the visible tools. Furthermore they were informed that they could ask for support on technical matters (how to handle
GeoGebra). In situations where students did not know how to proceed, the author encouraged them to explain what they had done and why they thought that their strategies did not work. When students felt they had solved the task they were asked why they were convinced they had come to a solution. Data was captured through screen recording, with integrated voice and video recording.

**Analysis method**

Research question 1 concerns the relationship between the students' reasoning and the feedback generated by GeoGebra. The students' reasoning was categorized using Lithner’s framework of creative and imitative reasoning (2008). The way that students used GeoGebra’s feedback was examined using the concepts verificative and elaborative feedback (Shute, 2008). The relationship between the students’ reasoning and GeoGebra’s feedback was analyzed by considering whether the students’ way of reasoning before and after a computer activity could be related to their way of using the feedback from GeoGebra. Research Question 2 concerns how the results from RQ1 relate to students' problem solving success. This will be analyzed by considering whether important decisions are consequences of certain reasoning and the use of feedback from GeoGebra. The analysis methods indicated here will be elaborated in the following text.

The data consisting of conversations, computer interactions, and gestures was transcribed into written text. In order to discuss the students’ reasoning and their way of using feedback from GeoGebra in relation to their success in problem solving the eight pairs were divided into two groups; those who reached a reasonable solution and those who did not.

Schoenfeld’s protocol-analysis provides a way to examine the way students’ decisions shaped the way that solutions evolved (Schoenfeld, 1985, p.292). In order to structure data the transcripts were partitioned into episodes according to Schoenfeld’s six phases of problem solving, i.e. reading, analyzing, planning implementing, exploring, and verification. Thereafter possible decision points were identified, i.e. junctions between episodes, occasions where new information arose from computer activities or students’ discussions, and sequences accompanied by difficulties. Actual decisions, when students’ utterances or activities indicate how to proceed were noted. These parts were used to consider in what way the decisions contributed to solving parts of the task and if information gained from solving parts of the task were used to answer the main question of the task.

Lithner’s (2008) framework was used to classify students’ reasoning into IR or CMR. The students’ conversations, interaction with GeoGebra and gestures were examined and units of argumentation were identified. The characteristics of the argumentation, i.e. the implicit or explicit justifications of the strategy choices and the strategy implementations, were used to determine if the reasoning fulfilled the characterizations of imitative or creative reasoning (Lithner, 2008). The students’ reasoning was regarded as CMR if there were signs of creating a (for the students) new
method of solving the problem and if their argumentation was anchored in intrinsic mathematical concepts. The reasoning was categorized as imitative reasoning if the (sub) task solutions were based on familiar facts and/or procedures.

Finally, the way students used GeoGebra’s feedback was examined using the concepts verificative and elaborative feedback (Shute, 2008). Dialogues and gestures before and after each computer activity were noted. A computer activity in this study includes the student input and the outcome displayed by GeoGebra. Before this moment the students will plan (planning phase) what to submit to the software and afterwards the students may interpret the outcome and discuss how to proceed (verificative and analytic phase). An utterance in a planning phase when the students predicted the outcome of a computer activity was interpreted as preparation for using the information from GeoGebra as verifying feedback. After a computer activity, in the verificative phase, students could use the feedback from GeoGebra verifiably, identified as utterances of success or failure. If they, after the verification, used the information to explain, plan how to proceed with the task solving, etc. they were considered to be, using the information from the software elaborately, entering the analytic phase. Finally, the situations of preparing activities and using feedback from GeoGebra were compared to whether the reasoning were classified as either CMR or IR.

To answer RQ1, the use of feedback, verifiably and/or elaborately was associated to the characteristics of reasoning, IR or CMR during the planning of the activity, and to the reasoning when using feedback.

To answer RQ2 the reason for students’ success or failure in solving the task were related to decisions that the students made and could have made. It was then considered whether the success or failure was related to the characteristics of reasoning and to the use of feedback.

RESULTS

Six of eight pairs came to a reasonable solution for the main task. They used possible decision points for solving sub problems, and used gained information to solve the following sub problems and the main task. Two pairs did not reach a reasonable solution to the main task. Two pairs started out as the less successful pairs but changed strategy and completed the task as the more successful pairs. Since none of pairs had a clear understanding of the formula $y=mx+c$ they all needed to clarify the properties of the formula. The following example is from such a sequence and will exemplify analysis of reasoning and utilization of feedback.

Episodes and decision points

During their attempt at finding a solution, S1 and S2 went through episodes of reading, exploring, planning/implementation, analyzing, and verifying. They had possible decision points at the junctions of episodes and when the computer activity generated new information. Two of those decision points particularly supported their problem
solving. The first of these decision points emerged when they realized that they did not fully understand the formula $y=mx+c$, and they decided to analyze the properties of the formula. The second decision point arose when they had difficulties locating a perpendicular function to $y=7x-1$, and they decided to change the function to $y=2x-1$ since (2) is easier to divide than (7). It was also clear that they used information from these episodes of analysis later in the task solving process. In the next paragraph their first episodes of exploring will be analyzed.

**Reasoning**

After reading the instructions they initiated an exploring episode as follows:

1. S2: well let’s just submit something...
2. S1: y is equal to seven....
3. S2: That means it’s going to be very much like this (pointing almost vertically, in front of the screen)

The suggestion to choose seven as the $x$-coefficient is followed by a prediction of the graphical appearance on the screen. They created the strategy themselves and S2’s utterance and gesture is interpreted as predictive argumentation. This strategy of suggesting something followed by a prediction of the result supported by argumentation, reappeared several times during their work. Some predictions were followed up by verificative argumentation, e.g. “$m=7$ means the line must increase by 7 every step to the right”, or “this one must have $m$ less than 1 because you go more steps horizontally than vertically”. Their reasoning is classified as CMR.

**Feedback**

The following excerpt, considered as an episode of analysis, will exemplify the way S1 and S2 used the information after submitting the function $y=-3x-1$, which they predicted to have “negative but less slope than $y=7x-1$”:

1. S1: This is not 90°....
2. S2: No it’s not... but let’s measure it to see how far off we are [uses GeoGebra's angle tool to measure the angle]...

After a discussion resulting in the conclusion that the constant term does not affect the slope of the function and that the slope depends only on $m$, the $x$-coefficient:

3. S1: we must concentrate on $m$....

After an analysis of different examples of submitted functions S1 summarized using $y=2x-1$ and $y=7x-1$ as references:

1. S1: Well, if we start at minus one.... This one has $m=2$.... Then you go one step to the right and then two upwards [counting squares with the mouse].... And this has $m=7$... if you go one step to the right you go seven upwards [counting squares with the mouse]....

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First they used the GeoGebra’s feedback for verification, concluding that they did not have a perpendicular line, and then they initiated an attempt to elaborate on the result. This led them to an episode of analysis where they elaborated on the feedback and investigated the way m and c affect the graphical representation. During their work these students frequently discussed and elaborated on the feedback they received according to which they adjusted their strategies. This indicates that they were using feedback from the software both as verificative and elaborative feedback.

**Relations between reasoning, feedback, and success in problem solving**

S1 and S2 frequently used CMR to predict the outcome of the computer activities, and they used the feedback from GeoGebra both for verification and elaboration. Furthermore, these students always related their elaborations to their predictions. This indicates relationships between CMR and elaboration on feedback from GeoGebra. It seems that predictions of computer activities that are founded in CMR gives ground for elaborately using the feedback that is received.

S1’s and S2’s decisions to examine the formula $y=mx+c$ and to replace the $x$-coefficient of (7) with (2) are considered as important for solving the task. Both decisions were taken in episodes of analysis and preceded by elaboration on feedback based on CMR. Information from analysis was then used to answer the main question of the task. These students’ engagement in CMR, and their elaborative use of feedback in the episodes of analysis seems important for their success in solving the task. Examples of students that are not reasoning creatively and not using feedback elaborately are provided in the following paragraph.

Examples of imitative reasoning and only verifying utilization of feedback

This extract is from two students as is considered as exploring the circumstances for the task:

1. S3: c was where it intersects the y-axis…
2. S4: yes…
3. S3: yes it was… but what is m…
4. S4: m was that value in between…
5. S3: yes… the difference when you go…
6. S4: yes…
7. S3: eh… what should I write then…

The utterance on line 1 and 3, and the attempts to explain on line 4 and 5 indicate that these students are trying to remember the way $c$ affects the intersection to the $y$-axis and the way to calculate the $x$-coefficient. The articulated facts are not coherent and there is no argumentation. This is characteristic for imitative reasoning. In the next excerpt there is an example of using information only verifying. The students have
submitted the function $y = -2x - 2$ for the purpose of creating a perpendicular line to $y = 2x - 2$:

1. S3: no… that is not perpendicular… it is too large… submit $y = -1x - 2$

2. S4: [submits $y = -1x - 2$]… this is not 90°…

3. S3: no… but we can change this one (pointing at $y = 2x - 2$) into $y = 1x - 2$…

Instead of analyzing why $y = -2x - 2$ did not appear perpendicular to $y = 2x - 2$ they changed the values of the two functions until they had perpendicular lines. They used the information as verifying feedback indicating if they were wrong or right but they did not elaborate on the feedback.

**DISCUSSION**

The results in this study are illuminated by the perspective of imitative and creative reasoning as well as the students’ way of using information from the software as feedback. Still there are unanswered questions about why they reason and use information as feedback in certain ways. Some of the explanations may be in the way the task is designed and some in the way GeoGebra invites interaction. What is not investigated in the study is the students’ attitudes towards mathematics and mathematics education. Anyhow, more or less implicitly there is an assumption in the reports of Hiebert and Grouws (2007), Lithner (2008), and Schoenfeld (1985) that students are guided into the attitude that mathematics education is about a teacher telling one how to solve certain task and that one is supposed to remember these methods and procedures and recall them when needed. The approach of student S3 and S4 might be due to teaching that essentially consisted of instructions and tasks that were prepared to solve as per the instructions, as well as student S1 and S2 may have experienced problem-oriented teaching. On the other hand Di Martino and Zan (2010) found that student attitudes are not stable and often change over time due to age, environment, topics, and activities. Di Martino and Zan suggest teachers should identify the students’ “attitude-profile”, in particular the significance for negative attitude associated to mathematics difficulties. Then the teachers can act purposefully to change the attitude of the student. It is paramount to make students aware of the importance of attitude, giving good examples, and identify success. Di Martino and Zan also point out the importance of adapting teaching towards activities focusing on processes rather than short questions and short answers. In summary, it is reasonable to assume that some of the differences regarding the students’ success solving the task in the study can be attributed to their attitudes toward mathematics, as well as the fact that this activity is suitable to work with students’ attitudes as well as their mathematical conceptualization.

**References**


The results presented in this paper are from an ethnographic case study of seven novice primary school mathematics teachers’ professional identity development. During their teacher education, these novice teachers experienced a new way of teaching mathematics. At the time of their graduation, they expressed a wish to change how mathematics is taught in schools. However, they also expressed that practicing teachers might prevent them from implementing this change because practicing teachers might want to stick to traditional methods. Two years after graduation, the respondents had not succeeded in implementing the changes they wanted to the teaching of mathematics. By following the respondents in the two years after their graduation, the study shows how practicing teachers become a limitation for the respondents, not by interfering but by being absent. The title of the paper, Just as expected and exactly the opposite, reflects this change.

INTRODUCTION
The starting point for this paper is a Swedish study aimed at understanding and describing the professional identity development of novice primary school teachers of mathematics (Palmér, 2013). The teaching profession, whether or not it focuses on mathematics, is often described as a changing profession where the roles and responsibilities of teachers have extended and broadened in past years and where there is little continuity between teacher education and practice (Cooney, 2001; Ensor, 2001; Fransson & Gustavsson, 2008; OECD, 2005; Sowder, 2007). This is because teacher education offers, “or almost provokes” (Bjerneby Häll, 2006, p.204), changes of perspective in student teachers as they, by soft coercion, are led into the predominant “style of thought” (Persson, 2009, p.143) at their university. Further, there is very little systematic coordination between teacher education and novice teachers starting in the profession in several European countries (Fransson & Gustavsson, 2008).

The novice teachers in this study work as primary school teachers, teaching several subjects, of which mathematics is one. This is similar to other countries around the world where most primary school teachers are educated as generalists (Tatto, Lerman
At the time of their graduation, the novice teachers expressed a wish to change how mathematics is taught in schools. During their teacher education, they experienced a new way of teaching mathematics in line with what is often called the reform (Ross, McDougall & Hogaboam-Gray, 2002; NCTM, 1991 & 2000). This wish is similar to what several other studies have shown (for example, Bjerneby Häll, 2006; Cooney, 2001; Sowder, 2007). However, the novice teachers also expressed several limitations they thought would prevent them from teaching mathematics according to methods and procedures outlined in the reform. In this paper, one of these limitations, i.e. practicing teachers, will be focused on. At the time of their graduation, the novice teachers expressed that practicing teachers would probably limit their possibilities to move mathematics teaching in a reformative direction since practicing teachers would want to stick to tradition. This paper will describe how the novice teachers dealt with this expected limitation in the two years after their graduation and how it influenced their professional identity development.

THEORETICAL FRAMING

Graduating from teacher education and starting to work as a teacher can be understood as a transfer or shift in professional identity. In research of professional identity development the interplay between the individual and their social environment is of importance (McNally, Blake, Corbin & Gray, 2008). Studies show that school culture, colleagues, parents, students and school leadership influence the professional identity development of novice teachers. That is why understanding the social environment of novice teachers is of great importance (Day, Kington, Stobart & Sammons, 2006; McNally, Blake, Corbin and Gray 2008).

In the study presented in this paper, this understanding (the interplay between the individual and their social environment) is developed through investigating the novice teachers’ participation and reification in communities of practices (Wenger 1998). The analysis of the novice teachers’ professional identity development and their expectations of being limited by practicing teachers will be based on the novice teachers’ memberships in communities of practice.

According to Wenger (1998), identity development is to be understood as the negotiated experience of self in the learning trajectory within and between communities of practice. A community of practice is defined through the three dimensions of mutual engagement, joint enterprise and shared repertoire. Mutual engagement is the relationships between the members, it is about them doing things together as well as negotiating the meaning within the community of practice. Joint enterprise is the mutual accountability the members feel in relation to the community of practice and it is built by mutual engagement. The shared repertoire in a community of practice is its collective stories, artefacts, notions and actions as reifications of the mutual engagement.

Memberships in communities of practice affect our identity in various ways. An individual can participate in communities of practice through engagement, alignment
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and/or imagination. Engagement implies active involvement and requires the possibility to physically participate in activities. Participation through alignment implies that the individual changes, aligns, in relation to the community of practice the individual wants, or is forced, to join. Imagination implies going beyond time and space in a physical sense and creating images of the world, making it possible to feel connected even to people we have never met but who, in some way, match our own patterns of actions. All three kinds of memberships are constantly changing, and learning trajectories in communities of practice can be peripheral, inbound, inside, on the boundary or outbound (Wenger, 1998).

METHOD

When wanting to acquire a deep understanding of a phenomenon, Flyvbjerg (2006) advocates choosing a few cases where the respondents have maximum experience of what is to be investigated. In this study, seven novice teachers were followed for two years after graduating from teacher education.

The teacher education the respondents graduated from comprised 210 credits. These 210 credits consisted of three integrated parts: a general educational area (90 credits including 15 credits student teaching, orientation (60-90 credits including at least 15 credits student teaching and two specialisations (60 credits). Orientation relates to subjects or subject areas and is chosen by the student teachers before starting teacher education. The respondents in this study were selected because they chose mathematics as one of their main subjects and some of them also wrote their final teacher education Bachelor theses on mathematics education. The average age of the respondents (all female) at the time of graduation was 31 years.

According to Winsløw et al. (2009) “there is a huge variation, among systems of education, when it comes to official regulations for the institutional transition from university to school – if they exist at all” (p.98). At the time of this study, there was no national or local teacher induction in Sweden, and after graduation the respondents were to seek employment on their own.

An ethnographic approach was used to make visible the interplay between the individuals and their social environment. Ethnography implies looking at, listening to and thinking about social phenomena with the main interest to understand the meaning activities have for individuals and how individuals understand themselves and others (Arvatson & Ehn 2009; Aspers 2007; Hammersley & Atkinson 2007).

The empirical material in this study is from observations, interviews and self-recordings made by the respondents. All of these were done in a selective intermittent way (Jeffrey & Troman 2004), which means that the time from the start to the end of the fieldwork was long, but with a flexible frequency of field visits. The observations were whole day observations, both participatory and non-participatory, attending lessons, the teachers’ staff room, meetings, etc. During observation, field notes were taken and written up later the same evening. Photos were taken and transcripts from
lessons were occasionally e-mailed to the respondents for comments. The interviews consisted of both spontaneous conversations during observation and formal interviews (individual and in groups) based on thematic interview guides. The respondents used mp3-players for their self-recordings. They were told to record whatever and whenever they wanted and that it was up to them to decide what was important for the researcher to know about starting work as a primary school teacher of mathematics. By allowing the respondents to choose what and when to record, an opportunity was opened to discover what they focused on and discerned in their day-to-day work. As such, the recordings guided what to talk about in interviews and what to look at in observations.

In the analysis, these varying empirical materials were treated as complete-empiricism (Aspers, 2007). The analysis was done using two different, but cooperating, ways using communities of practice as a lens and methods inspired by grounded theory. Methods inspired by grounded theory have been used to encode the empirical material. Using grounded theory for encoding implies building and connecting categories grounded in the empirical material by using codes (Charmaz, 2006). In the study, one such category was frames for teaching mathematics. This category was developed through coding segments where the respondents, on their own initiative, expressed (words and/or actions) obstacles, difficulties and/or resistance regarding their mathematics teaching. Within this category, the limitation being focussed on in this paper, i.e. practicing teachers is a subset.

RESULTS

The results are presented in three sub-sections. In the first sub-section, the time of graduation is focused on. The second sub-section contains a summary of the two years after graduation. Finally, in the third sub-section, segments from a group interview two years after graduation are presented. The joint theme in all three sub-sections is the similarities found in the unique experiences and expressions of the respondents regarding practicing teachers.

The empirical examples presented are not to be seen as the wholeness of what the category frames for teaching mathematics is based on, but as examples of empirical instances labelled as obstacles, difficulties and/or resistance related to practicing teachers within that category. The empirical examples are from self-recordings and interviews since these are possible to reproduce word-for-word. However, in the describing text and also in the analysis the observations compose an important part. Since the aim is to focus on a subset of a category – practicing teachers as a frame for mathematics teaching – and not on any specific respondent, no names are mentioned in relation to the empirical examples. Instead, as previously stated, the empirical examples are to be understood as examples of empirical instances labelled as obstacles, difficulties and/or resistance related to practicing teachers within that category. As such the empirical examples represent observed and spoken experiences of all respondents.
The time of graduation

The first time the respondents were interviewed was just before graduating from teacher education. At the time of graduation, all respondents expressed a clear opinion regarding how mathematics ought to be taught in primary school. They said that they had encountered a “new way” of teaching mathematics in their teacher education and now they wanted to “reform” mathematics teaching in schools. This new way of teaching mathematics differed from the mathematics teaching they themselves had experienced as pupils and during their teaching practice within teacher education.

Before graduation, the respondents expressed an expectation that this “new way” of teaching mathematics might not be compatible with mathematics teaching in schools. Several times they talked about a need to “fit in” and when asked what that implied, they, for example, said:

You know, you have that picture of a teacher. How a teacher is supposed to be and how a teacher is supposed to act. And as a new teacher you may have to yield a little to your big ideas. [...] Yes, I have, you need to have backing for what you want and what you want to do. And then I think, it can be hard to suggest new ideas. [...] I can come up with a lots of ideas but they might say ‘What are you doing now? ’ Or ‘What’s happening? We don’t work like that. (Example from interview)

Even if the respondents had a clear picture of how they wanted to teach mathematics after graduation, they expressed insecurity regarding how this mathematics teaching would be received in “reality”. They expressed that practicing teachers might become a limitation for their possibilities to teach mathematics as they would like to.

You can’t bring too much that is new in the beginning either, you have to sneak it in a little, if it is old and deep-rooted. (Example from interview)

The two years after graduation

During the time of this study, it was difficult for primary school teachers in Sweden to get jobs, especially in certain municipalities. After graduation, the respondents started to work at different schools and preschools as class teachers, long-term and short-term substitute teachers and as teacher assistants. Some of them frequently taught mathematics, others taught sporadically. However, in one way or another, all of them taught mathematics during the two first years after graduation. From those two years, the empirical material consists of observations, interviews and self-recordings. What is common for all the respondents is a lack of evidence of using the “new way” of teaching mathematics that they had emphasised before graduation.

During these two years the respondents often express that they lack opportunities to collaborate with and get support from other teachers. According to the respondents, other teachers at the schools do not take much notice of them during the two years
after graduation. They say that nobody is nasty but that everyone has too much to do and, therefore, does not have time for others.

There is absolutely no organisation on the whole at the school. And no, it is a little different. Not different but everyone is running their own race. […] nobody knows what I do. I can do what I want to because nobody [cares]. There are no goals and no matrixes. Nothing in common. (Example from interview)

The respondents express this loneliness as something negative since “new teachers need to be taken care of in the beginning”.

[…] it would have felt good for me as completely new if she had told me that [that she is doing a good job] and been engaged. Showed that she cares about me. (Example from interview)

Also those who work as short-term substitute teachers say they are lonely even though everybody is nice to them.

So, you do not make any connections, like deep connections. […] Yes, it is lonely. You have to take care of yourself a lot. (Example from interview)

When the respondents have the opportunity to visit or to collaborate with other teachers they emphasise these experiences as important.

This morning I was with Mark at his Swedish lesson. I feel that it is a great privilege to be with an older colleague who has worked with writing for a long time. […] It’s really educational. […] It’s really wonderful to see how the older teachers somehow have the ability. It’s really educational. (Example from self-recording)

**Group interview two years after graduation**

Two years after graduating from teacher education, the respondents were gathered for a group interview. In this interview, they were asked what, two years after graduation, they thought about their teacher education in general and their teacher education in mathematics education in particular. Even though they were not completely happy with their teacher education as a whole, they spoke positively about the mathematics components it had contained. They still talk similarly as before graduation about how they would like to teach mathematics in a “new way”.

When talking about their teacher education and their wish to change mathematics teaching they spontaneously start to talk about their wish to change mathematics teaching and why they have not managed to do this so far. And, just like they did before graduation, they start to talk about practicing teachers as a limitation. However, not because practicing teachers have prevented the novice teachers from teaching mathematics as they want to, but as a limitation by being absent. Before graduation the respondents thought that they would have to adapt to the prevailing method of teaching mathematics to be accepted. Two years after graduation they
instead express that the lack of opportunity to collaborate with and get support from
other teachers have prevented them from teaching mathematics as they want to.

I feel that the thing you need when you get out there is a mentor. [...] You
would almost need to work as a trainee, in that sense. (Example from group
interview)

The worry the respondents expressed before graduation regarding being limited by
practicing teachers had changed into a wish that other teachers would take more
notice of them and help them just like they expressed during the two years after
graduation. Two years after graduation, other teachers are still expressed as a
limitation, but not by preventing the respondents from teaching mathematics as they
want to, but by being absent. Teaching mathematics, as they wanted was not easy
because, as one of them says, “I can do whatever I want to do but what do I do then?”

ANALYSIS

In this section, the analysis of the novice teachers’ professional identity development
and the expected limitation of practicing teachers is based on the novice teachers’
memberships in communities of practice.

At the time of graduation, the novice teachers were outbound members through
engagement and imagination in a community of reform mathematics teaching. The
core of this community of practice is located in the teacher education that the
respondents have no contact with after graduating. In this community of practice,
there is a shared repertoire regarding how mathematics ought to be taught in a “new
way”. The respondents express that this “new way” of teaching mathematics differs
from the mathematics teaching they themselves have experienced at school, both as
pupils and student teachers. As such, they have experiences of other ways of teaching
mathematics but they do not want to teach mathematics like that. This aversion seems
to be a part of the shared repertoire in the community of reform mathematics
teaching. At the time of graduation, the respondents have an expectation that
practicing teachers will limit their possibilities to change mathematics teaching in
line with the shared repertoire in the community of reform mathematics teaching.
This expectation seems to be a reification in the shared repertoire of the community
of reform mathematics teaching.

The two years after graduation are very different for the respondents but they all
work as teachers in one way or another. What they have in common is a feeling of
loneliness since, contrary to what they had expected, there is no cooperation between
teachers in the schools. The novice teachers work alone and not much of the kind of
mathematics teaching they talked about before graduation is visible. As the
respondents do not cooperate with other teachers, they do not develop new
memberships in communities of practice regarding mathematics teaching. As such,
they are not influenced by other shared repertoires regarding mathematics teaching
and they do not contribute to the development of new ones. Nor do they deepen their membership in the community of reform mathematics teaching. In fact, it is quite the opposite, as the core of that community of practice is located within the teacher education with which the respondents have no contact after graduation.

Two years after graduation, the respondents’ professional identities as primary school mathematics teachers are very different but they still speak similarly about how they would like to teach mathematics. Two years after graduation, they are peripheral members in the community of reform mathematics teaching. Since they do not have any physical interaction with other members of this community of practice after graduating from teacher education this membership is mainly through imagination. Even if they do not express any objections toward the reform mathematics teaching in its shared repertoire, they do not teach mathematics in line with it.

The lack of opportunities to collaborate with other teachers prevents the respondents from developing new memberships in communities of (mathematics) teaching which make their professional identity development lonely. Two years after graduation, the respondents express that the absence of cooperation with practicing teachers prevent them from teaching mathematics in line with the shared repertoire in the community of reform mathematics teaching. The respondents wish that other teachers would take more notice of them. The title of the paper, *Just as expected and exactly the opposite*, reflects this change. Other teachers became a limitation, not by interfering, but by being absent.

**DISCUSSION**

At the same time as the results and analysis in this paper illuminate some issues regarding novice teachers’ expectations and experiences, they also raise new questions that require further investigation. One question regards the community of reform mathematics teaching located within the teacher education of these respondents. The shared repertoire in this community of practice seems to be in line with the reform (Ross, McDougall & Hogaboam-Gray, 2002; NCTM, 1991 & 2000). As mentioned in the introduction of this paper, novice teachers’ commitment to this kind of mathematics teaching has been shown in several other studies, and is therefore not remarkable. What is remarkable in this study is the respondents’ expectation of being limited by practicing teachers. Why, at the time of their graduation, did the novice teachers expect other teachers to be a limitation rather than a support?

The first two years after graduation were very varied for the respondents. They worked at different schools and preschools as class teachers, long-term and short-term substitute teachers and as teacher assistants. However, regardless of these variations, they experienced the same loneliness. Is the teaching profession a lonely one, or were these novice teachers simply unlucky? At the time of their graduation, the respondents expected that practicing teachers would limit their possibilities to
teach mathematics as they would like to. While they were not limited by practicing teachers, they expressed the lack of cooperation as a limitation instead. This contrary situation raises a question regarding what kind of cooperation with other teachers the respondents did expect?

Finally, perhaps the most important question remains unanswered. Why is it that these novice teachers did not manage to teach mathematics in line with the shared repertoire in the community of reform mathematics teaching? From the perspective of the respondents themselves, the answer is the absence of cooperation with other teachers. However, from an outside perspective, simply any kind of cooperation with other teachers does not seem to be a realistic solution. Based on the lack of cooperation, a teacher induction seems to be needed. However, not one focused on personal comfort, which teacher induction programs historically have focused on (Wang, Odell & Schwille, 2008). Instead, novice teachers seem to be in need of cooperation regarding (mathematics) teaching. This is in line with results in an anthology of research regarding novice teachers in northern Europe (Fransson & Gustavsson, 2008). In this anthology, it is proposed that, instead of acting as a safety net, teacher induction should focus on content knowledge and pedagogical content knowledge. However, if such a teacher induction would result in outcomes other than those presented in this paper is yet to be investigated.

REFERENCES


TEACHER CHANGE VIA PARTICIPATION TO A RESEARCH PROJECT
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Our focus here is teacher change that was one of the objectives of the research project at hand. In this paper only a brief description of the longitudinal research project that was implemented in 2010–13 in the Department of Teacher Education at the University of Helsinki, is given. The main data is based on the interviews of two voluntary teachers and the videos taken from their mathematical problem-solving lessons. During the project the researchers noticed a change in experimenting teachers’ attitudes towards open problem solving, and more generally toward the open teaching of mathematics.

THE RESEARCH PROJECT

The three-year study ran in the Department of Teacher Education at the University of Helsinki and was financed by the Academy of Finland for the years 2010–13 (the project 135556). The objective of the project was to clarify the development of pupils’ and teachers’ mathematical understanding and problem solving skills over the course of three years – from grade 3 to grade 5 – when open problems were used regularly once a month. It was a joint comparative research project with Chile where we aimed to parallel the Finnish and Chilean teaching practices in mathematics. This paper is limited to the Finnish portion of the project.

Since this paper will concentrate on one part of the project (teacher change) and due to the fact that the project has been dealt with elsewhere (e.g. Pehkonen & al. 2013, Pehkonen 2014), this paper will only provide the main features of the project which concern the development of teachers’ implementing methods of open-ended problems in their teaching group. Thus, the research question is as follows:

How does teachers’ pedagogical content knowledge develop when they use open-ended problems in mathematics teaching?

Implementation of the study

In the main project, there were two groups involved in the study. Both groups were comprised of roughly 10 teachers: the experimental group and the control group. The former group was from the surrounding cities of Helsinki (Vantaa, Espoo, Kirkkonummi) and the latter was from Helsinki. In the experimental group, the teachers allocated, on average, one mathematics lesson per month to concentrate on open-ended problems, whereas the teachers of the control group applied conventional methods for mathematics teaching. Data was gathered through a variety of methods which included questionnaires that were answered by both pupils and teachers, drawings made by the pupils, teacher interviews, classroom observations and field notes that were collected during the implementation phase, video of the pupils working on the open problems, thinking-aloud protocols and discussions which were videotaped.
It was not expected that such a short inference in teaching (an open problem lesson only once a month, with all of the other lessons taught conventionally) would make a substantial change, however we anticipated that once teachers and pupils had experienced open mathematics teaching (open problem solving), they would gradually change their understanding of mathematics and its teaching.

The experimental tasks used in the project, were open problems in which additional options were available to the pupil at either the initial or final stages of the problem. Solving these problems required that the solver must combine information already familiar to him/her in a new way. These tasks were introduced beforehand in the teachers’ and the researchers’ monthly joint meetings. In the meeting the tasks were given proper wording and a method of presentation was elaborated. Every teacher, then pondered the implementation of the task in her own teaching group, and gave the researcher her lesson plan before the experimental lesson. Altogether 20 different tasks that dealt with various mathematical topics were covered. The topics included arithmetic combinatorics and geometry.

TEACHERS’ PROFESSIONAL GROWTH

In this paper we will focus on the professional growth of two experimental teachers. The answers to this question had been clarified by interviewing these teachers and reviewing videotaped recordings of their lessons that span a two year period. A preliminary report has been written on Finnish experimental teachers’ professional growth (cf. Portaankorva-Koivisto & al. 2013, 2014). According to previous findings we noticed that these two teachers emphasized different aspects in their teaching and the tasks got unexpected configurations in their hands. We also found that during the project these teachers learned how to organize and implement problem-solving lessons. Additionally, the teachers seem to have gained insight into the importance of the beginning and ending of the process. In this paper we conceptualize these findings further and let the teachers narrate their experiences. Firstly we will deal with some theoretical aspects of the teachers’ professional growth and connect that growth to the use of open-ended problems.

How to guide pupils’ problem-solving process

Ainley and Luntley (2007) offer a novel theoretical framework that they call a teacher's attention-dependent knowledge. According to them the performance of experienced teachers consists of an interplay between their subject knowledge and pedagogical knowledge, but their attention-dependent knowledge about their pupils is revealed in the classroom. According to Ainley and Luntley, a teacher’s activities during the lesson reflect in what way the teacher is able to recognize or interpret the mathematical situations in the classroom. If we observe the teacher and the class events we can note, for example, whether the teacher identifies pupils’ mathematical discourse, or the way the teacher organizes to support her pupils’ mathematics learning in phases, and how well does she identify her pupils' learning problems.
According to a constructivist learning approach, the teacher must take into account her pupils' prior knowledge, and to support the pupils’ personal relevance of their own learning. The teacher should help her pupils to tolerate uncertainty, encourage them to question the teacher's pedagogical plans and methods, to take responsibility for their own learning and to develop their negotiation skills (cf. Taylor, Fraser & Fisher, 1997). In the development of these skills the use of open problems offers the teacher a better chance to accomplish these goals than ordinary mathematical tasks.

Even though, teachers may know the benefits of open problems and appreciate them, they might feel uncertainty about using them. Open problems may cause uncertainty, because they challenge both the teacher’s content knowledge and her pedagogical content knowledge (Applebaum & Leikin, 2007). Here, the teacher’s content knowledge (cf. Shulman, 1986) is comprised of her own understanding of the essence of mathematical challenge, her knowledge of challenging mathematics, and her ability to approach challenging tasks. Moreover, teachers’ pedagogical content knowledge includes knowledge of how pupils cope with challenging mathematics, as well as knowledge of appropriate learning settings (Applebaum & Leikin, 2007).

Consequently, some teachers tend to avoid open problems, perhaps, because they are afraid of new and unfamiliar situations. If the teacher is encouraged to use open problems in her mathematics teaching, it might, however, happen that she turns these problems into less open problems. This can happen if the teacher’s understanding of openness is different from that of the researcher’s. Thus, the teacher changes the task according to her understanding of what is meant by openness. (Tzur, 2008, Swan, 2007).

We can structure teacher’s actions, when she is conducting a problem-solving situation in her class, with the help of the teaching model for problem solving developed by Laine & al. (2013) and is based on the well-known Polya model. The teaching model has four steps: (0) Planning the lesson, (1) The task introduction, (2) The guidance phase, and (3) Summary and feedback.

In the planning phase the teacher may change the role of an open problem, because it does not, in her opinion, correspond to her teaching objectives. Teachers seem to prefer mathematical routine tasks to problems, because problems may be too difficult for the pupils (and also for teachers), and the pupils might lack the necessary background information. Hence teachers often postpone problems, in the introduction, to the stage, where they take shape rather as routine tasks. Often, procedures and formulas have already been taught during the previous lessons and the teacher tries to actively recall them during the introduction phase before turning to the present problem (cf. Tzur, 2008; Swan, 2007).

Problems which are dealt with may also be changed during the guidance phase, if the teacher determines that the pupils are not progressing as desired. The teacher may notice pupils' poor motivation or the teaching group may be unnecessary restless. Then the teacher stops the work in class, in order to give more instructions and guide pupils.
towards proper mathematical thinking. Every now and then, the teacher shortens pupils’ working time, so that peace will be restored. Sometimes the teacher notices that pupils are not able to solve the problem, and she decides to teach something to help pupils to proceed, or she directs the work by giving a clue to the entire class (cf. Tzur, 2008; Swan, 2007). Thus, during implementation open problems might change their shape and it follows a kind of learning other than the purpose for which they were initially intended.

**Teacher’s view of mathematics and open problem-solving**

When we focus on developing mathematics teaching by using open tasks the teacher's own view of mathematics, meaning, her understanding of what mathematics is as well as what constitutes good teaching, heavily influences her teaching methods. Open problem solving requires the teacher possess not only good content knowledge and pedagogical skills, but also a more advanced view of mathematics, as well as the ability to teach in a flexible manner. When these occur, the teacher will be able to listen to her pupils, flexibly adjust her teaching according the needs of the pupils, and to share responsibility while working with the pupils.

The teacher's weak content knowledge often inhibits or even becomes an insurmountable barrier to the use of open problems. Recently some researchers such as Häkkinen & al. (2011) have suggested that if we want to reach mathematical problem-solving goals in our basic education, we have to focus on future primary teachers’ content knowledge even before they begin their studies in teacher education. They suggest that future teachers’ mathematics tests should be examined more thoroughly to determine, if the errors they have made were meaningless, as small calculation errors, or do errors based on erroneous logic that reveals that the basic concepts are not at all understood or that strong deficiencies are present in their content knowledge. Additionally, the strengthening of experienced teachers’ content knowledge will remain one of the goals of in-service teacher education.

**Teachers’ view of teaching and teaching methods**

The methods the teacher selects to use during her mathematics lessons are usually linked to the teacher’s view of teaching. Patrikainen (2012) in her doctoral dissertation describes following three teaching conceptions. She suggests that the teacher's teaching and actions can be viewed on the basis of these conceptions.

According to the humanistic-constructivist view of teaching, the teacher's primary goal is to create a safe and motivating learning environment for the pupils. The teacher cherishes methods that motivate and activate pupils. She selects meaningful activities and seeks to take notice of the diversity of her pupils in her teaching. In accordance with the cognitive-constructivist view of teaching, the teacher focuses on building a solid knowledge base. In this case, when the teacher is planning the teaching-studying-learning process she designs it in such a way that it is based on the previously learned content, and the topic will be addressed in as many ways as possible. The contextual-constructivist view of teaching emphasizes understanding in mathematics and the
cumulative origin of mathematical knowledge. In this case, the teacher often has to juggle between teacher-centredness and pupil-centredness, in order to adhere to her view of teaching. (Patrikainen, 2012)

In terms of the problem-solving process each of these views of teaching also has, in spite of the constraints, positive aspects. The humanistic-constructivist teacher can create an open, creative and open-minded atmosphere that supports the problem-solving process. By means of open problems the cognitive-constructivist teacher can develop pupils’ mathematical knowledge base. Moreover, the contextual-constructivist teacher can increase pupils’ knowledge of mathematical applicability.

The study at hand is focused on two primary teachers and their development of their teaching methods with regard to open problems, i.e. the way they controlled and guided the mathematical problem-solving process – pedagogical content knowledge (cf. Shulman, 1986). Furthermore, we describe these two primary teachers’ method of controlling and guiding pupils’ open problem solving as well as to reflect on the factors that influence the teacher’s actions throughout the process. We try to detect links between teachers’ views of teaching and their actions in the teaching group based on the typology constructed by Patrikainen (2012).

**Teachers’ interviews**

This paper will concentrate on two teachers (Ann and Beate) who volunteered to be interviewed. The data consists of interviews conducted by the second author in the spring of 2012 and 2013. Field notes were taken during the lessons, and these were used to support the interview data. In addition, the earlier videotaped problem-solving lessons that both teachers had collected during the academic years of 2010-2011 and 2011-2012 were analysed. This paper refers to the following two open problems.

*Flags with stripes* (grade 3, January 2011): Design flags with three stripes and three colors. Design as many flags as possible. Draw your solutions.

*Arithmagon* (grade 3, February 2011): Arithmagon is a triangle in which there are numbers marked on the three sides; the task is to find numbers to be placed at the corners so that each pair of corner numbers will add up to the side number between them (e.g. Mason 1982, 196). Find the corner numbers that can fit in the triangle (see Figure 1).

![Figure 1. An example of Arithmagon triangles (left), and the task given to the pupils (right).](image-url)
The data analysis was carried out using theory-driven content analysis (cf. Yin, 2014). The interviews were transcribed and analysed concentrating, in particular, on the teachers’ answers about mathematics, its learning and teaching, as well as mathematical problem solving. Those answers were mirrored in activities and choices the teachers had selected for their problem-solving lessons and also their guiding process.

**Participating teachers**

Teacher Ann has been a teacher for about 20 years. Ann attended our Finland–Chile research project following her school principal’s proposal, but she was happy to participate. She wanted to learn about the different ways of teaching mathematics, and get some variety in her pupils’ classwork.

Teacher Beate has been a teacher for about 10 years. She explained that while the school principal had brought an invitation to participate in the project, she immediately became excited because of her interests in Chile and the Spanish language. The project also offered a possibility to learn to teach problem solving, which is something that Beate was not very familiar with at that time. She also stressed the importance of the fact that participation in the project was a unique experience for her pupils.

These two teachers were both having a profound teaching experience, and they also shared an interest in mathematics, but when it comes to their actions in the class and their teaching styles, they were very different. For example, in the paper of Laine & al. (2013) it is shown that these two teachers varied significantly in terms of posing questions during the problem-solving lesson.

**RESULTS**

Both teachers’ activities in class reveal that they have made progress in interpreting and listening pupils’ mathematical situations (cf. Ainley & Luntley, 2007).

“*I have observed that more time should be given to pupils’ thinking ... earlier I thought that the textbook was more important than I think it is today.*” (Beate, interview, 2013)

“*I often ask the children to help and ask them how they would explain this to their friend.*” (Ann, interview, 2013)

Teachers, however, interpreted the situations relying, rather, on their knowledge of their pupils instead of relying on their subject knowledge or pedagogical content knowledge.

Problems dealt with changed even in the hands of these teachers. Ann's deficiencies in mathematics limited her ability to guide problem solving in a successful direction, nevertheless she has thrown herself into problem solving and also given her pupils the possibility to do so as well.
"Your flags are all different. There are different colors and you have used every kind of shape and pattern: triangles, stars and circles. But three colors, good!" (Ann, in *Flags with stripes*, grade 3, January 2011)

Ann's primary goal was to create a safe and motivating learning environment for her pupils in such a way that there was always room for creativity, too. She cherished the experiential methods that she affirmed to be motivating and activating, and she emphasized meaningful activities, and the pupils’ autonomy.

"Interacting with pupils means being present [and] the attunement is an important part of the lesson." (Ann, interview, 2012)

Ann's perception of teaching can be classified as humanistic-constructivist (cf. Patrikainen, 2012).

Beate was so precise with her educational aims that she connected the problems to topics pupils have already learned and unknowingly made them easier. She was aware of the fact that the pupils had enough previous knowledge and assisted the pupils in recalling that knowledge.

"We are about to start a problem-solving lesson and I have drawn an aritmagon on the blackboard. We already discussed aritmagons during our previous lesson. Does anyone have an idea of how we can solve it?" (Beate, in *Arithmagon*, grade 3, February 2011)

She prepared her lessons with accuracy.

“Design as many flags as you can. The flags need to be striped, and each flag has to have three colors: green, yellow and purple. ... I have helped you a little bit. Here you have a worksheet with rectangles already there so that it is easy to start to design and colour your flags.” (Beate, in *Flags with stripes*, grade 3, January 2011)

Her goals reflected the fact that Beate wanted all her pupils reach a successful outcome, and she changed the plan if they were not progressing in the desired manner (cf. Tzur, 2008, Swan, 2007).

"How many of you are able to calculate backwards? What if you know the answer that should be in your aritmagon, but what you don’t know is how it has been achieved." (Beate, in *Arithmagon*, grade 3, February 2011)

On one hand Beate wanted to focus on building a solid knowledge base and plans the teaching-studying-learning process so, that it was based on the previously learned content and all of the topics were covered, in a structured way, as completely as possible.

“If you have planned the lesson in advance, you listen to your pupils differently and you will hear and understand their ideas more easily.” (Beate, interview, 2012)
On the other hand Beate emphasized that mathematics could be found all around one in one’s everyday life. She stressed the importance of understanding, and had the opinion that mathematical knowledge is cumulative. Thus, Beate had to juggle between teacher-centredness and pupil-centredness, she changed into teacher-centred practices as soon as her pupils’ solutions turned in an undesirable direction. Beate’s view of teaching contained both cognitive-constructivist and contextual-constructivist features of teaching (cf. Patrikainen, 2012).

**FINAL CONSIDERATIONS**

During the three-year research project both teachers became familiar with the elements of open problems in mathematics teaching. They gained knowledge of how to introduce the task, how to use ready-made templates, or the use of checkpoints in between. They began to understand the importance of the final summary, and their pupils' self-assessment, or the specific role of the pupils' feedback. But this professional development directs their attention to guide the pupils’ work, and as a side effect also reduces the openness of the tasks. On one hand the pupils' findings and solutions increase in quality, but also more uniform. On the other hand the weaknesses in the teachers’ content knowledge is reveled.

One of the research project’s objectives was to provide the participating teachers opportunities to develop their pedagogical knowledge; plan, try, make mistakes, and also to change their own conceptions of learning, while simultaneously relying on their experience and professional skills. Furthermore, they learned different solutions for implication of open problems from other teachers who participated in the experiment.

**A comment**

When dealing regularly (monthly) with open problems one may observe the development of positive attitude in pupils as well as their increased skills in problem solving and creativity. Some experimental teachers have committed to such teaching more than the others, but one may observe in all experimental teachers changes in the direction of innovative teaching. Such teaching reflects also on their pupils and could be seen in their problem solving. In order for teachers to be able to develop such a change in their teaching style, they should be interested in the development their own teaching and committed to finding a new solution (cf. Shaw & al. 1991).

Along with constructivism we became aware of teachers’ mathematics-related beliefs and their meanings. Here beliefs are understood as knowledge and feelings that are mainly based on earlier experiences (e.g. Pehkonen 1991). These beliefs conduct and structure every aspect of the teaching and learning process. In order for the teaching–learning process to be changed, teachers’ beliefs on good and successful teaching should be developed and changed.

In the literature there are many research reports on the requirements for teachers’ changing and developing processes (e.g. Shaw & al. 1991, Pehkonen 2007). The change seems to be especially challenging, if the teacher is alone (cf. Palmer 2013; also Chapman 2002). But none of the intervention methods, described so far, seem to be...
automatically successful. It appears that teachers can only be given ideas and directions regarding development possibilities. But they must themselves adapt those ideas and implement them with their own understanding, and until then they have them as a part of their teaching reservoir.

References


WHAT IS NOTICED IN STUDENTS’ MATHEMATICAL TEXTS?

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The study aims to identify the conceptions that are indicated in a teacher group’s discussions on a number of mathematical texts produced by students. The study uses the concept of noticing and the components ‘attending to’ and ‘interpreting’ elements in students’ mathematical texts. The preliminary findings suggest that teachers attend to what the students have done as well as to what the students have not done. Discussions on the former are related to students thinking processes whereas the latter is related students’ communication. An ideal mathematical text is implied and its features, often found to be ‘missing’ from the students’ texts, include explicit explanations or justification of calculations as well as, what is referred to as, mathematical language.

INTRODUCTION

Teaching mathematics involves creating and organizing learning situations in which mathematics can be explored and communicated orally and in writing. Teachers are expected to assess students’ mathematical competence using a number of different sources of information of which students’ mathematical writing is one example. What can be learnt about students’ mathematical competence from their mathematical texts depends on a number of factors. One important factor among these is what is being noticed and recognized by the teacher (Jacobs, Lamb, & Philipp, 2010). As stated by Kress (2009), “Only what is recognised and accorded full recognition as means and modes for learning can be assessed. What is not recognised will not and cannot be assessed.” (Kress, p. 38).

This paper examines teachers’ professional noticing when discussing students’ mathematical texts. Schoenfeld (2011) states that “teachers’ decision making is shaped by what teachers notice” (Schoenfeld, 2011, p. 233). It is thus important for teachers, teacher educators and researchers to know something about what it is that teachers notice. In the selection on what is being noticed and how this is interpreted, a teacher will have conceptions, both conscious and subconscious, about why something is ‘noticeable’. As Schoenfeld (2011) argues, “teachers’ decision making – of which noticing is a critical component – is a function of their resources, goals, and orientations (Schoenfeld, 2011, p. 231). As an affective part of this, we have conceptions. In this paper, we define conception as "a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preferences” (Philipp, 2007, p. 259). This would imply that conceptions may have both affective and cognitive dimensions. Here, our first aim is to study teachers’ conceptions as in terms of what they notice from an affective point of view when discussing students’ written mathematical texts. The second aim is to test
noticing as a theoretical tool when studying teachers’ conceptions. The research questions are the following:

- What elements from students’ mathematical texts are noticed and subsequently discussed by teachers and how are these elements interpreted?
- What conceptions are indicated by these interpretations?

**BACKGROUND**

This paper focuses on students-written mathematical communication and teachers’ noticing. There are few studies looking at written mathematical communication and teachers’ responses to it. One of the few is Morgan (1998, 2001) who has examined teachers’ ideas on what constitutes “good” writing in mathematics. Her research suggests that teachers have very vague ideas on why one student’s mathematical text would be superior to another. They have difficulty articulating their own criteria for “good” communicative skills. Morgan (1998) identifies, in research on writing in mathematics, a general assumption that the act of writing and the process of interpreting and judging students’ writing are unproblematic and that mathematical texts produced by students are seen to provide transparent records of students’ intentions as well as their comprehension and cognitive processes.

It could be argued that students’ mathematical writing in school typically serves two very different functions (Morgan, 1998). It can be seen as a part of a learning process in which writing is used to record and perhaps reflect on various mathematical ideas; hence, the text is written by and for the student herself. It can also, however, be seen as a product for the purpose of assessment; hence, written for a teacher or examiner. Unlike the work of professional mathematicians, the work in school mathematics often serves these two functions at the same time.

Noticing is a complex activity that is influenced by an individual’s knowledge, experience and beliefs (Star, Lynch, & Perova, 2011). Researchers have found distinct patterns of noticing for groups with similar goals and experiences such as professional groups. Goodwin (1994) refers to this as *professional vision* and provides an example of an archeologist and a farmer who will see “quite different phenomena in the same patch of dirt” (1994, p. 606). The idea of a particular expertise of a group of professionals has attracted attention in many different fields (Jacobs, Philipp, & Sherin, 2011). In mathematics Mason (2002) introduced the discipline of noticing as a way for teachers to sensitize themselves to notice opportunities in their practice. Following this Jacobs et al. (2010) investigated teachers professional noticing of children’s mathematical understanding and developed a framework for unpacking the in-the-moment decision making of teachers in everyday classroom situations involving children’s verbal or written mathematical communication. They suggested three components to this skill, attending to children’s strategies, interpreting children’s mathematical understandings and deciding how to respond on the basis of children’s understanding. They applied their framework to interviews with different groups of
teachers who were presented with video recordings of classroom situations as well as written material produced by students. They found that different groups of teachers with varying amount of experience and expertise indeed not only attend to different features in classroom situations and in written material but also display different levels of depth in their interpretations. 

Despite using diverse conceptualizations of the concept of noticing most researchers focusing on the concept of noticing in mathematics education agree that noticing involve two main processes:

- Attending to particular events in an instructional setting
- Making sense of events in an instructional setting (Sherin, 2011)

Sherin (2011) notes that these processes are interrelated and cyclical and that the concept of noticing implies dynamic rather than static knowledge.

**METHOD**

Nineteen elementary school teachers from four schools in a middle-sized town in mid-Sweden were interviewed in groups. There were five groups of 3, 4 or 5 teachers respectively. At the time of the interview all nineteen teachers were teaching mathematics. They were initially chosen by their principals and asked to participate based on their own interest. The interviews were all recorded on video as well as with an additional audio recorder. The teachers were presented with 10-16 mathematical texts produced during a problem-solving activity by 10-11 year old students from three different classes. The problem-solving was centered on two specific problems which are presented below in two examples of mathematical texts from the data. The teachers were given information on the problems but very little information on the situation in which the texts were created. Being faced with a mathematical text whose origin the teacher knows very little about forces a teacher to focus on the text itself and the interpretations are derived to a larger extent from the text than they would have been had the teacher been asked to comment on their own students’ written material. The teachers were asked to discuss the different texts from an assessment perspective with a particular focus on the way they students communicated their ideas through different modes (for example drawings, symbols and words). They were asked to comment on what they considered to be “good” and “less good” examples of mathematical texts and to provide arguments for their reasoning. A group interview was chosen so as to create room for discussions but also for eliciting the teachers’ idea of a ‘common ground’ in evaluating the students’ texts.
Although five groups were interviewed, in this paper the focus is on the discussions one four-teacher group had regarding the student-produced mathematical texts.

The arguments and reasoning provided by this group was analyzed by focusing on the two main processes of noticing, that is, *attending to* and *interpreting* as described above with reference to Sherin (2011). The analysis was guided by three questions: (1) What elements are teachers noticing in the mathematical texts?; (2) How are teachers interpreting these elements? (3) How can similar or different interpretations be categorized?

The interview (55 minutes) was transcribed. In the transcription verbal language markers such as shortening of words and vernacular use of different tenses were replaced with more or less grammatically correct versions. This ‘washing’ of the language was done with consideration taken to the level of detail in the following analysis. The analysis was done in three steps. The first step involved identifying data carrying information on what the teachers turned their attention to in the texts. This data was then analyzed with the intent of identifying the teachers’ interpretations. The second step of the analysis was concerned with identifying similarities/differences in the interpretations of the teachers. This part of the analysis follows the structure of thematic analysis as described by Braun and Clarke (2006). When the two categories of interpretations, presented in the findings, had been created the analysis turned to the third step, which involved the conceptions, as defined above that were indicated by the identified interpretations. These conceptions were analyzed by looking for presuppositions in the categories. This could for example be to use the quote “They are not using words to describe their thought process” [T1] to elicit the presupposition *it is possible, for these students, to use words to describe their thought processes*. The results that are presented below are preliminary.
RESULTS

The teachers in the interview notice a number of elements from the students’ texts in their discussions. They attend to different parts of the texts such as drawings, calculations, narratives, statements and erased material but they also view the text as a whole. Their interpretations of the different elements are related to the perceived thinking processes and actions of the students but also to the students’ communication. One way of categorising what the teachers notice and how they interpret this was to introduce the two categories: (1) what the students have done; and, (2) what the students have not done. These two categories are very distinct with very different characteristics and they appear throughout the interview.

When the teachers discuss what the students have done they spend considerable time trying to identify the students’ problem-solving strategies. It seems important for the teachers to correctly identify these strategies and they are attempting to come to agreement with regard to what strategy the students seem to have used. They offer suggestions on a possible strategy by posing a question so as to look for confirmation from the others e.g. “She has gathered sort of four legs… creates an animal with one head… is that it?” [T1]. They also present arguments to support their suggestions e.g. “She has probably drawn all the circles first…and then she has probably erased some because here she has made two and two… more twos” [T2]. The teachers seem to use all available information when trying to understand the student’s strategies, including erased material. Erasing produces traces and these can be used to infer the use of a particular strategy.

When the teachers attend to what the students have not done they focus their discussion on what is ‘missing’ in the students’ mathematical texts. In these discussions the interest in group agreement is less apparent.

Below are quotes that highlight the two different categories:
<table>
<thead>
<tr>
<th>What the students are doing</th>
<th>What the students are not doing</th>
</tr>
</thead>
<tbody>
<tr>
<td>He has counted how many fours he has removed and those are plus signs… it is plus 7 [T3]</td>
<td>It doesn’t say anything about the motorcycles there… [T3]</td>
</tr>
<tr>
<td>She is drawing a little bit… she is sort of drawing the problem… making a picture of it… [T1]</td>
<td>There is no real investigation… here you can ask… where did you get that four from? [T2]</td>
</tr>
<tr>
<td>… it shows that they have erased, right?… 5 or whatever it said before… they have used trial-and-error [T4]</td>
<td>How did they get it to be 3… I don’t really understand the picture… how did they get it to be 3 cats and 9 hens? [T2]</td>
</tr>
<tr>
<td>She has used trial-and-error… since it has been erased… [T3]</td>
<td>This girl doesn’t have any mathematical language whatsoever that she has shown in her solution [T1]</td>
</tr>
<tr>
<td>Yes, she is grouping them then? [T1]</td>
<td>They haven’t used the equals sign correctly [T2]</td>
</tr>
</tbody>
</table>

The teachers’ comments on what the students have done tend to relate to the students’ processes (whether as actions or as thinking processes). The teachers’ discussions on what the students have not done seem to relate to the students’ communication. This indicates that teachers are viewing the students’ processes the intent of understanding those processes, whereas the students’ communication is viewed from a deficiency perspective. When the teachers discuss what is ‘missing’ from a text they imply that there is an ‘ideal’ solution to which they compare the students’ texts. From the discussions it is possible to sketch some features of this ideal solution. An ideal mathematical text includes; information on that which is not explicitly asked for, an explicit explanation or justification of how the numbers in the calculations were derived, mathematical language and correct use of mathematical conventions. In most, but not all, instances in which this ideal mathematical text is discussed it also seems that this ideal can be expected of fourth grade students.

The conceptions that are indicated by the discussions are thus;

- It is very important to correctly identify a student’s problem-solving strategy
- There is an ideal mathematical text, to which all mathematical texts are measured. The features of this ideal are only implied but discernible. Most of the features of the ideal text can be expected of students in grade four.
DISCUSSION

The teachers in the interview deal with the positive, what is there, and the negative, what is not there although it should be. When dealing with what ‘is’ there the teachers seem to consider everything, even erased material, in trying to understand what the students have done and this supports the idea that this is very important for them. It is possible that the teachers consider this to be one of their most significant responsibilities.

When teachers deal with what is not there in the students’ texts they seem to focus on the students’ communicational skills rather than the mathematical content. The analysis suggests that there are unarticulated ideas on an ideal mathematical text that are connected to the teachers’ expectations regarding the students’ writing. Identifying and highlighting the features of this ideal should be important for the purpose of both teaching and assessment for both teachers and students. The sociomathematical norms that guide ideas such as the ‘ideal mathematical text’ are rooted in what Schoenfeld (2011) refers to as teachers’ goals, orientations and resources. In this study this concerns the teachers’ conceptions about mathematics as a subject as well as their conceptions on learning and assessment in mathematics. When teachers talk about what is missing in students’ mathematical texts, they are not mentioning the mathematical understanding but rather features that are important for communicational reasons. A certain level of communication seems to be necessary for teachers to be able to assess the students’ other mathematical skills and understanding. This indicates that teachers do differentiate between the two functions of mathematical writing that Morgan (1998) identified: the personal when the students writes for herself and the public when the students writes for a teacher. That which is written for a teacher should display certain features. These features are in turn indicators of certain skills or understandings. In this respect the findings are consistent with Morgan’s study although the teachers in this study were rather vague about the reasons for wanting particular features to be included in a mathematical text. Possible implications of this vagueness and the implicitness of the ideal mathematical text are that communicational skills may be seen as innate and static rather than a characteristic that can be developed.

As was found by Jacobs et al (2010) teachers can change their way of noticing different aspects of their practice as they gain more experience. Our conclusion is that noticing as a theoretical tool is useful, given that knowledge about what it is that teachers notice and value in student-written mathematical texts, and why they value this, provide a common ground for vital discussions on assessment in mathematics in professional development as well as pre-service training.
References


One of the key domains to be taught in preschool is geometry, specifically, identifying two-dimensional figures and using mathematical language to describe those figures. This paper investigates teachers’ knowledge and self-efficacy for identifying and defining triangles and circles. Results indicated that in general, teachers had a high self-efficacy for both identifying and defining triangles and circles, were able to identify both triangles and circles, were able to define triangles, but had some difficulty defining circles.

In Israel, as in other countries, the preschool curriculum encourages teachers to engage children with geometrical activities with the aims of having children identify and name basic shapes as well as use mathematical language to describe these shapes (INMPC, 2008; NCTM, 2006). In order for teachers to carry out such activities they must be knowledgeable of the subject matter as well as believe that they have the knowledge required to teach geometry. In a previous paper we reported on preschool teachers’ self-efficacy and knowledge related to identifying triangles and circles (Tirosh, Tsamir, Levenson, Tabach, & Barkai, 2011). Taking into consideration the importance of language and definitions in mathematics, this paper expands on that study by investigating preschool teachers’ self-efficacy and knowledge for defining triangles and circles, as well as their self-efficacy for identifying triangles and circles.

THEORETICAL BACKGROUND

There are basically two main issues at hand in this paper. The first relates to learning and teaching geometry and the second relates to teachers’ knowledge and self-efficacy. These issues are reviewed briefly below.

Much of young children’s geometrical knowledge is based on their perceptions of their surroundings. Later on, examples serve as a basis for both perceptible and non-perceptible attributes, ultimately leading to a concept based on its defining features. Such a process was described by Vinner and Hershkowitz (1980) who introduced the terms concept image and concept definition in reference to geometrical concepts. A concept image includes all associations with the concept, including verbal, visual, vocal, and possibly other sensory associations as well. Van Hiele (1958) theorized that students’ geometrical thinking progresses through a hierarchy of levels, and that at the most basic level, students use visual reasoning, taking in the whole shape without considering that the shape is made up of separate components.

This study was supported by the Israel Science Foundation (grant No. 654/10).
At the next van Hiele level students begin to notice different attributes of different shapes. Attributes may be critical or non-critical (Hershkowitz, 1989). In mathematics, critical attributes stem from the concept definition. However, these same concepts may have been encountered by the individual in other forms prior to being formally defined, invoking concept images which may or may not coincide with the definition.

One of our major aims as educators is to bring our students to use only definitions as the deciding factor in identifying examples and forming geometrical concepts. The concept definition refers to "a form of words used to specify that concept" (Tall & Vinner, 1981, p. 152). A formal concept definition is a definition accepted by the mathematical community whereas a personal concept definition may be formed by the individual and change with time and circumstance. Using precise mathematical language may contribute towards the aim of having students refer to concept definitions. Thus, teachers, including preschool teachers, must not only be able to identify various examples and non-examples of geometrical figures, but they should be knowledgeable of mathematical definitions, the critical attributes of each geometrical figure, and the mathematical language that is used when discussing these figures (Ball, Hill, & Bass, 2005).

Teachers' self-efficacy is another factor related to teachers' classroom behaviour. Hackett and Betz (1989) defined mathematics self-efficacy as, “a situational or problem-specific assessment of an individual’s confidence in her or his ability to successfully perform or accomplish a particular [mathematics] task or problem” (p.262). Several studies which reported on preschool teachers' self-efficacy investigated teachers' beliefs regarding their general ability to teach and not their specific ability to teach mathematics (e.g., McMullen, 1997). In one study (Bates, Latham, & Kim, 2011), early-childhood prospective teachers’ mathematics teaching self-efficacy was investigated by analyzing teachers' responses to general statements such as “I will continually find better ways to teach mathematics”. However, levels of self-efficacy are not necessarily equal in all domains and tasks and there is a need to devise instruments and interventions that will address the specificity of teachers’ self-efficacy (Tsamir, Tirosh, Levenson, Tabach, & Barkai, 2015).

This study takes into consideration the interrelationship between knowledge and beliefs which can affect teachers’ proficiency (Schoenfeld and Kilpatrick, 2008) and is guided by the Cognitive Affective Mathematics Teacher Education (CAMTE) framework (e.g., Tirosh et al., 2011). The framework, which has also been used to guide professional development for preschool teachers, is presented in Table 1. In Cells 1-4, and in Cells 5-8, we address teachers' knowledge and self-efficacy respectively.
Table 1: The Cognitive Affective Mathematics Teacher Education Framework

<table>
<thead>
<tr>
<th>Knowledge</th>
<th>Pedagogical-content</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solving</td>
<td>Evaluating</td>
</tr>
<tr>
<td>Cell 1:</td>
<td>Cell 2:</td>
</tr>
<tr>
<td>Producing</td>
<td>Evaluating</td>
</tr>
<tr>
<td>solutions</td>
<td>solutions</td>
</tr>
<tr>
<td></td>
<td>Knowledge of students’ conceptions</td>
</tr>
<tr>
<td></td>
<td>Designing and evaluating tasks</td>
</tr>
<tr>
<td>Self-efficacy</td>
<td>Cell 3:</td>
</tr>
<tr>
<td>Cell 5:</td>
<td>Cell 6:</td>
</tr>
<tr>
<td>Mathematics self-efficacy related to producing solutions</td>
<td>Mathematics self-efficacy related to evaluating solutions</td>
</tr>
<tr>
<td></td>
<td>Cell 7:</td>
</tr>
<tr>
<td></td>
<td>Pedagogical-mathematics self-efficacy related to children’s conceptions</td>
</tr>
<tr>
<td></td>
<td>Cell 8:</td>
</tr>
<tr>
<td></td>
<td>Pedagogical-mathematics self-efficacy related to designing and evaluating tasks</td>
</tr>
</tbody>
</table>

In previous studies we focused on Cells 1 and 5 and Cell 8 with regard to preschool teachers’ ability to identify various two and three-dimensional figures and their self-efficacy for designing tasks that may promote and evaluate children’s geometric knowledge (e.g. Tirosh et al. 2011; Tsamir et al. 2015). This study investigates preschool teachers' knowledge and self-efficacy regarding two different tasks – identifying figures and defining figures – and two different figures – triangles and circles (Cells 1 and 5). Specifically, we ask: (1) Are preschool teachers able to define a triangle and a circle and identify various examples and non-examples of triangles and circles? (2) What are preschool teachers' self-efficacy beliefs regarding their ability to define and identify triangles and circles? (3) What are the differences and relationships between teachers' knowledge and self-efficacy for defining and identifying triangles and circles?

METHOD

For the past several years, we have been providing professional development for preschool teachers aimed at promoting their knowledge for teaching mathematics in preschool (e.g., Tirosh, Tsamir, & Levenson, Tabach, & Barkai, 2011). The mathematical content of these programs varied according to the curriculum set by the Ministry of Education and according to the duration of the program. The main data for this study was gathered from a group of 19 teachers, teaching 4-6 year old children in municipal preschools. All had a first degree in education, specializing in early childhood education, which included at least two courses focusing on numeracy and geometrical concepts for young children.

At the beginning of the program, teachers were asked to complete a questionnaire with three parts: self-efficacy statements, defining figures, identifying figures. The first part of the questionnaire began with the following self-efficacy related questions: I am able to define a triangle. If I am shown a triangle, I will be able to identify it as a triangle. If I am shown a figure which is not a triangle, I will be able to identify it as not being a triangle. This was repeated for circles. A four-point Likert scale was used to rate participants’ agreements with self-efficacy...
statements: 1 – I do not agree that I am capable; 2 – I somewhat agree that I am capable; 3 – I agree that I am capable; 4 – I strongly agree that I am capable.

The definitions part of each questionnaire began with a request to define a triangle and a circle. Teachers were explicitly told to write a definition that would be mathematically acceptable and not necessarily one they would use when talking to children. After teachers completed the definitions, they submitted it to the didactician who then gave them the identifications part of the questionnaire. This part consisted of a series of figures (see Figures 1 and 2). Each figure was accompanied by a question: Is this a triangle (or circle)? Yes/No. It is important to note that an additional group of practicing teachers with a similar educational background were given only the second part of the questionnaire to answer. Thus, the number of teachers who were requested to give definitions was greater than the number of teachers who answered the other sections of the questionnaires.

Triangles and circles are figures explicitly mentioned in the mandatory Israel national mathematics preschool curriculum (INMPC, 2008). We did not include rectangles and squares, also mentioned in the curriculum, because we did not want to involve the complexity of hierarchy. In choosing the figures to present to the teachers, both mathematical and psycho-didactical dimensions were considered. That is, we not only consider whether the figure is an example or a non-example of a geometrical shape but what developmental and cognitive issues might arise as children identify geometrical figures. Specifically, we consider whether or not the example or non-example would intuitively be recognized as such. When considering triangles, for example, the equilateral triangle may be considered a prototypical triangle and thus intuitively recognized as a triangle, accepted immediately without the feeling that justification is required (Tsamir, Tirosh, & Levenson, 2008). The scalene triangle may be considered a non-intuitive example because of its “skinniness”. Whereas a circle may be considered an intuitive non-example of a triangle, the pizza-like “triangle” may be considered a non-intuitive non-example because of visual similarity to a prototypical triangle (Tsamir, Tirosh, & Levenson, 2008).
Is this a triangle? | Intuitive | Non-intuitive
--- | --- | ---
Examples | Equilateral triangle | Scalene triangle
Non-examples | Rounded-corner “triangle” | Pizza | Elongated pentagon | Open ”triangle”

Figure 1: Is this a triangle?

Is this a circle? | Intuitive | Non-intuitive
--- | --- | ---
Examples | Circle | 
Non-examples | Triangle | Spiral | Decagon | Ellipse

Figure 2: Is this a circle?

Triangles may vary in the degree of their angles providing a wide variety of examples. In contrast, the circle’s symmetry limits variability. Thus, only one example of a circle was given. The non-examples of each shape were also chosen in order to negate different critical attributes of each shape.

**RESULTS**

**Self-efficacy**

In general, the teachers (N=19) had a relatively high self-efficacy for both identifying and defining triangles and circles (see Table 2) with the lowest self-efficacy for being able to define circles.
Identifying Defining

<table>
<thead>
<tr>
<th></th>
<th>Identifying</th>
<th></th>
<th>Defining</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>SD</td>
<td>Mean</td>
<td>SD</td>
</tr>
<tr>
<td>Triangle</td>
<td>3.82</td>
<td>.38</td>
<td>3.74</td>
<td>.45</td>
</tr>
<tr>
<td>Circle</td>
<td>3.34</td>
<td>.58</td>
<td>3.31</td>
<td>.82</td>
</tr>
</tbody>
</table>

Table 2: Mean self-efficacy for identifying and defining triangles and circles

In order to further analyze the results, paired-samples t-tests were carried out (Table 3). Differentiating between triangles and circles, we see that teachers had a significantly lower self-efficacy for defining circles than for defining triangles. Likewise, teachers had a significantly lower self-efficacy for identifying circles than for identifying triangles. However, when it came to the task of identifying figures versus defining figures, no significant differences were found for either figure.

<table>
<thead>
<tr>
<th>Triangles versus circles</th>
<th>Mean difference</th>
<th>t-value</th>
<th>df</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Defining triangles versus defining circles</td>
<td>.42</td>
<td>2.65</td>
<td>18</td>
<td>.02</td>
</tr>
<tr>
<td>Identifying triangles versus identifying circles</td>
<td>.47</td>
<td>3.51</td>
<td>18</td>
<td>.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Identifying versus defining</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identifying triangles versus defining triangles</td>
</tr>
<tr>
<td>Identifying circles versus defining circles</td>
</tr>
</tbody>
</table>

Table 3: Comparing self-efficacy: different figures and different activities

**Triangle Identifications and Definitions**

Eighteen teachers completed all three parts of the questionnaire. An additional 13 preschool teachers answered only the definitions part of the questionnaire, resulting in 31 definitions for triangles. We begin by describing results of identifications and continue with teachers’ definitions.

Most teachers succeeded in identifying the examples and non-examples presented on the questionnaire (M=.96, SD=.14). Three teachers did not identify the rounded-corner "triangle" (See Figure 1) as a non-example. One teacher did not identify correctly the open "triangle" and one teacher did not identify correctly the elongated pentagon as a non-example. Interestingly, all incorrect identifications related to claiming that a non-example is an example of a triangle.

According to the Israel Ministry of Education's website, a triangle is defined as “a polygon with three sides.” This is a minimal definition. Out of 31 definitions, 29
were correct (five of those were minimal). One incorrect definition, "A triangle is a geometrical shape which has 3 straight lines", was missing the critical attribute of the shape being closed. The second incorrect definition, "A triangle is equal sided – it has 3 equal sides", narrows down the concept and excludes all triangles that are not equilateral triangles.

In addition to writing correct definitions for a triangle, teachers exhibited knowledge regarding various critical attributes for a triangle. The triangle has several critical attributes, including: polygon, closed, three sides, three vertices, three angles, the sum of every two sides is greater than the third, and the sum of the interior angles is 180 degrees. All of the teachers wrote the critical attribute of "three" (see Table 4). Perhaps because these teachers work with young children, few mentioned angles or referred to the sum of the angles. The notion of angle is usually not introduced until upper primary school and the sum of the angles of a triangle is usually derived from other properties during lower secondary school.

<table>
<thead>
<tr>
<th>Term</th>
<th>Frequency (%)</th>
<th>Term</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polygon</td>
<td>13 (42)</td>
<td>Shape</td>
<td>16 (52)</td>
</tr>
<tr>
<td>Closed</td>
<td>12 (39)</td>
<td>Three</td>
<td>31 (100)</td>
</tr>
<tr>
<td>Angles</td>
<td>9 (29)</td>
<td>Sides</td>
<td>24 (77)</td>
</tr>
<tr>
<td>Vertices</td>
<td>16 (52)</td>
<td>Corners</td>
<td>1 (3)</td>
</tr>
<tr>
<td>Straight lines</td>
<td>5 (16)</td>
<td>Straight sides</td>
<td>1 (3)</td>
</tr>
<tr>
<td>Angle sum 180°</td>
<td>2 (6)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Frequency (%) of terminology used in triangle definitions (N=31)

Circle Identifications and Definitions

In general, teachers (N=18) were able to identify the examples and non-examples presented (M=.98, SD=.07). One teacher incorrectly identified the ellipse as a circle and one teacher incorrectly identified the spiral as a circle.

An additional ten teachers were requested to write a definition of a circle, resulting in 28 potential definitions, however, only 23 teachers responded. An additional five statements related to the area and perimeter of a circle without defining a circle, leaving 18 statements to analyze.

According to the Israel Ministry of Education's website, a circle is defined as "the set of all points in a plane that are equidistant from a given point. That point is called the center of the circle." None of the teachers gave this exact definition or what may be considered an equivalent definition. Unlike the definitions given for a triangle, teachers' definitions for a circle were less clear and included a wide variety of terms. For example, one teacher wrote, "A closed shape, from the middle point go out equal rays." Thus, definitions were not categorized correct or
incorrect. Instead, we itemized the terms used according to their possible reference to the critical attributes of a circle. The circle definition given by the Ministry of Education leads to the following critical attributes: closed, planar figure (the set of all points), equidistant, given point. In the above example of a teacher's definition, it does not explicitly state "all points in a plane" but instead refers to a circle as a closed shape. We classified "middle point" as referring to some "given point". Although rays may not be measured, the teacher did write “equal” which we took as a reference to "equidistant". Table 5 displays the number of teachers who wrote terms which may be associated with critical attributes of the circle, including examples of the terms we linked with those attributes.

<table>
<thead>
<tr>
<th>Critical attributes</th>
<th>Closed figure/set of all points</th>
<th>Equidistant</th>
<th>Given point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Associated terms</td>
<td>Closed</td>
<td>Curve, shape</td>
<td>Middle point, center</td>
</tr>
<tr>
<td>Frequency</td>
<td>18 (100)</td>
<td>17 (94)</td>
<td>8 (44)</td>
</tr>
</tbody>
</table>

Table 5: Frequency (%) of terminology used in circle definitions (N=18)

Although none of the teachers specifically used the terms “planar figure” or “two-dimensional shape”, from our experience with teachers we concluded that the use of the term “shape”, for most teachers, refers to two-dimensional shapes. Nearly all of the teachers referred to the circles as a closed shape or curve, placing circles in the class of two-dimensional figures. Approximately half of the teachers referred to the notions of equidistance and having a given point from which this distance is measured. Two teachers, besides writing that the circle is a closed curve, added that the curve has no beginning and no end. This relates to that teacher’s concept image of a circle. A different teacher described the closed shape as being “round.” This description is related to both that teacher’s concept image and use of language. While other shapes, such as ellipses, may be considered round, in Hebrew, the term for round has the same root as the term for circle, so that the two concepts are linguistically linked. Vinner (2011) claimed that concept images include verbal associations. Similarly, the teachers who used the terms rays, instead of line segments, may not know the difference between rays and line segments. Or perhaps, for those teachers, a circle invokes an image of a point from where rays go forth in the sense that they have a beginning and go on from there, till they hit the circle itself. In other words, although the teachers may have used terms which were not always precise, they did use mathematical terms, and, except for the one teacher who described the circle as “round”, they did not revert to everyday language to define the circle.
Interestingly, four teachers wrote that a circle does not have certain attributes. For example, one teacher wrote, a circle is “a closed shape without any vertices.” While a shape that has vertices cannot be a circle, not all shapes that are without vertices, are circles. Another teacher wrote that a circle is, "a shape without vertices where all of the parts are distanced equally from the center." For these teachers, their image of a circle includes not only what it is, but what it is not, or what attributes it cannot have.

Comparing self-efficacy and knowledge

In general, as seen in Table 6, teachers had a high self-efficacy for identifying triangles and circles as well as a high score for correctly identifying both figures. On the other hand, as seen in Table 3 (above) teachers had a significantly lower self-efficacy for defining circles than for defining triangles, whereas their correct identifications of circles was quite high. In other words, their actual knowledge of identifying circles was higher than their perceived self-efficacy.

<table>
<thead>
<tr>
<th></th>
<th>Self-efficacy for identifying</th>
<th>Correct identification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>3.82</td>
<td>.96</td>
</tr>
<tr>
<td>Circle</td>
<td>3.34</td>
<td>.98</td>
</tr>
</tbody>
</table>

Table 6: Mean self-efficacy and identifications scores

Regarding self-efficacy for defining triangles and teachers’ actual definitions of triangles, a high self-efficacy for identifying triangles (M=3.74) corresponded to a high ability to correctly define triangles along with a broad and accurate knowledge of the critical attributes of a triangle. This correspondence was not duplicated for circles. Although teachers’ self-efficacy for defining circles was significantly lower than their self-efficacy for defining triangles, it was still relatively high (M=3.31). Yet, teachers’ definitions of circles were not on the same level as their triangle definitions and their knowledge of the critical attributes of a circle was partial. Thus, it may be said that while teachers were aware that their ability to define circles was lower than their ability to define triangles, they had a higher self-efficacy belief than was reflected in their actual definitions.

DISCUSSION

Knowledge and beliefs are interrelated and both should be considered when planning professional development for teachers (Tsamir, Tirosh, Levenson, Tabach, & Barkai, 2015). Teachers in this study had some difficulties when it came to defining circles. While it may be unwarranted to introduce young children to formal definitions, the teacher should be aware of mathematical definitions and be able to discuss with children the critical attributes of a concept which are derived from the definition. If a teacher limits her description of a circle to a figure which is round and not pointy, children may develop a concept image of a circle which includes ellipses. On the other hand, if teachers are aware that there exists...
a point from which all points on the perimeter of a circle are equidistant from that point, then they may plan activities that demonstrate this attribute of a circle. Teacher educators can take advantage of the fact that preschool teachers are able to define triangles and demonstrate to teachers the relationship between definitions, critical attributes, and identifying examples and non-examples. In turn, teachers may come to appreciate how knowing definitions for additional figures can then assist them in their endeavour to enrich children’s geometrical knowledge.

Regarding self-efficacy, for the most part, teachers were aware of their abilities for identifying and defining triangles and circles. Wheatley (2002) claimed that teachers’ efficacy doubts may cause a feeling of disequilibrium which in turn may foster teacher learning. Thus, teachers’ lower self-efficacy for defining circles may actually motivate them to learn more about circles. On the other hand, when comparing teachers’ self-efficacy for defining circles to their actual definitions, there seems to be some discrepancy. One reason for this dissonance could be that teachers had a clear concept image of circles and felt that this would enable them to define circles as well. Another reason could be that teachers equated definitions of geometrical figures with descriptions of geometrical figures, and thought that if they describe a circle it could be considered as a definition. Professional development for preschool teachers might take these results into consideration and not only promote the teachers knowledge of geometrical concepts but their knowledge of the nature of definitions in mathematics and the importance of language, especially for young children who are developing their concept image for these figures.

References


